

MOMENT-SEQUENCE TRANSFORMS

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ABSTRACT. We classify all functions which, when applied term by term, leave invariant the sequences of moments of positive measures on the real line. Rather unexpectedly, these functions are built of absolutely monotonic components, or reflections of them, with possible discontinuities at the endpoints. Even more surprising is the fact that functions preserving moments of three point masses must preserve moments of all measures. Our proofs exploit the semidefiniteness of the associated Hankel matrices and the complete monotonicity of the Laplace transforms of the underlying measures. We also examine transformers in the multivariable setting, which reveals a new class of piecewise absolutely monotonic functions.

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1. INTRODUCTION

The ubiquitous encoding of functions or measures into discrete entities, such as sampling data, Fourier coefficients, Taylor coefficients, moments, and Schur parameters, leads naturally to operating directly on the latter “spectra” rather than the original.

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The present article focuses on operations which leave invariant power moments of positive multivariable measures. To put our essay in historical perspective, we recall a few similar and inspiring instances.

The characterization of positivity preserving analytic operations on the spectrum of a self-adjoint matrix is due to Löwner in his groundbreaking article [22]. Motivated by the then novel theory of the Gelfand transform and the Wiener–Levy theorem, in the 1950s Helson, Kahane, Katznelson, and Rudin identified all real functions which preserve Fourier transforms of integrable functions or measures on abelian groups [15, 20, 26]. Roughly speaking, these Fourier transform preservers have to be analytic, or even absolutely monotonic. The absolute monotonicity conclusion was not new, and resonated with earlier work of Bochner [6] and Schoenberg [28] on positive definite functions on homogeneous spaces. Later on, this line of thought was continued by Horn in his doctoral dissertation [17]. These works all address the question of characterizing real functions F which have the property that the matrix $(F(a_{ij}))$ is positive semidefinite whenever (a_{ij}) is, possibly with some structure imposed on these matrices. Schoenberg’s and Horn’s theorems deal with all matrices, infinite and finite, respectively, while Rudin et al. deal with Toeplitz-type matrices via Herglotz’s theorem.

In this article, we focus on functions which preserve moment sequences of positive measures on Euclidean space, or, equivalently, in the one-variable case, functions which leave invariant positive semidefinite Hankel kernels. As we show, these moment preservers are quite rigid, with analyticity and absolute monotonicity again being present in a variety of combinations, especially when dealing with multivariable moments.

The first significant contribution below is the relaxation to a minimal set of conditions, which are very accessible numerically, that characterize the positive definite Hankel kernel transformers in one variable. Specifically, Schoenberg proved that a continuous map $F : (-1, 1) \rightarrow \mathbb{R}$ preserves positive semidefiniteness when applied to matrices of all dimensions, if and only if F is analytic and has positive Taylor coefficients [28]. Later on, Rudin was able to remove the continuity assumption [26]. In our first major result, we prove that a map $F : (-1, 1) \rightarrow \mathbb{R}$ preserves positive semidefiniteness of all matrices if and only if it preserves it on Hankel matrices. Even more surprisingly, a refined analysis reveals that preserving positivity on Hankel matrices of rank at most 3 already implies the same conclusion.

Our result can equivalently be stated in terms of preservers of moment sequences of positive measures. Thus we also characterize such preservers under various constraints on the support of the measures. Furthermore, we examine the analogous problem in higher dimensions. In this situation, extra work is required to compensate for the failure of Hamburger’s theorem in higher-dimensional Euclidean spaces.

We conclude by classifying transformers of tuples of moment sequences, from which a new concept of piecewise absolutely monotonic functions of several variables emerges. In particular, our results extend original theorems by Schoenberg and Rudin.

Besides the classical works cited above delineating this area of research, we rely in the sequel on Bernstein’s theory of absolutely monotone functions [4, 32], a related pioneering article by Lorch and Newman [21] and Carlson’s interpolation theorem for entire functions [8].

As a final remark, we note that entrywise transforms of moment sequences were previously studied in a particular setting motivated by infinite divisibility in probability

theory [18, 30]. The study of entrywise operations which leave invariant the cone of all positive matrices has also recently received renewed attention in the statistics literature, in connection to the analysis of big data. In that setting, functions are applied entrywise to correlation matrices to improve properties such as their conditioning, or to induce a Markov random field structure. The interested reader is referred to [3, 13, 14] and the references therein for more details.

Organization. The plan of the article is as follows. Section 2 recalls notation, reviews previous work, and lists our main results for classical positive Hankel matrices transformers. Sections 3 to 6 are devoted to proofs, arranged by the domains of the entries of the relevant Hankel matrices. Section 7 deals with multivariable transformers of Hankel kernels. Section 8 makes the natural link with Laplace transforms and interpolation of entire functions. Section 9 contains a summary, in tabular form, of the results proved in this article, as well as an index of our ad hoc notation.

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2. PRELIMINARIES AND ONE-VARIABLE RESULTS

Let μ be a non-negative measure on \mathbb{R} , with moments of all orders, and let its moment data and associated Hankel matrix be denoted as follows:

$$s_k(\mu) = s_k := \int_{\mathbb{R}} x^k d\mu, \quad \mathbf{s}(\mu) := (s_k(\mu))_{k \geq 0}, \quad H_\mu := \begin{pmatrix} s_0 & s_1 & s_2 & \cdots \\ s_1 & s_2 & s_3 & \cdots \\ s_2 & s_3 & s_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (2.1)$$

All measures appearing in this paper are taken to be non-negative and are assumed to have moments of all orders. We will henceforth call such measures *admissible*.

Throughout this paper, we allow matrices to be semi-infinite in both coordinates. We also identify without further comment the space of real sequences (s_0, s_1, \dots) and the corresponding Hankel matrices, as written in (2.1).

Central to our study is the class of *absolutely monotonic entire functions*. These are entire functions with non-negative Taylor coefficients at every point of $(0, \infty)$. Equivalently, it is sufficient for such a function to have non-negative Taylor coefficients at zero. Their structure was unveiled in a fundamental memoir by Bernstein [4]; see also Widder's book [32].

Definition 2.1. Given subsets $I, K \subset \mathbb{R}$, let $\text{Meas}^+(K)$ denote the admissible measures supported on K , and let $\mathcal{H}^+(I)$ denote the set of complex positive Hermitian semidefinite Hankel matrices with entries in I . We will henceforth use the adjective 'positive' to mean 'complex Hermitian positive semidefinite' when applied to matrices.

The following theorem combines classical results of Hamburger, Stieltjes, and Hausdorff; see, for instance, Akhiezer's book [1].

Theorem 2.2. A sequence $\mathbf{s} = (s_k)_{k=0}^\infty$ is a moment sequence for an admissible measure on \mathbb{R} if and only if the Hankel matrix with first column \mathbf{s} is positive. In other words, the map $\Psi : \mu \mapsto (s_k(\mu))_{k=0}^\infty$ is a surjection from $\text{Meas}^+(\mathbb{R})$ onto $\mathcal{H}^+(\mathbb{R})$. Moreover,

- (1) *restricted to $\text{Meas}^+([0, \infty))$, the map Ψ is a surjection onto the positive Hankel matrices with non-negative entries, such that removing the first column still yields a positive matrix;*
- (2) *restricted to $\text{Meas}^+([-1, 1])$, the map Ψ is a bijection onto the positive Hankel matrices with uniformly bounded entries;*
- (3) *restricted to $\text{Meas}^+([0, 1])$, the map Ψ is a bijection onto the positive Hankel matrices with uniformly bounded entries, such that removing the first column still yields a positive matrix.*

Definition 2.3. In view of the above correspondence, we denote by $\mathcal{M}(K)$ the set of moment sequences associated to measures in $\text{Meas}^+(K)$. Equivalently, $\mathcal{M}(K)$ is the collection of first columns of Hankel matrices associated to admissible measures supported on K . We write $H^{(1)}$ to denote the truncation of a matrix H in which the first column is excised.

Transformations which leave invariant Fourier transforms of various classes of measures on groups or homogeneous spaces were studied by many authors, including Schoenberg [28], Bochner [6], Helson, Kahane, Katznelson, and Rudin [15, 20]. From the latter works, Rudin extracted [26] an analysis of maps which preserve moment sequences for admissible measures on the torus; equivalently, these are functions which, when applied entrywise, leave invariant the cone of positive semidefinite Toeplitz matrices. Rudin's result, originally proved by Schoenberg [28] under a continuity assumption, is as follows.

Theorem 2.4 (Schoenberg, Rudin). *Given a function $F : (-1, 1) \rightarrow \mathbb{R}$, the following are equivalent.*

- (1) *Applied entrywise, F preserves positivity on the space of positive matrices with entries in $(-1, 1)$ of all dimensions.*
- (2) *Applied entrywise, F preserves positivity on the space of positive Toeplitz matrices with entries in $(-1, 1)$ of all dimensions.*
- (3) *The function F is real analytic on $(-1, 1)$ and absolutely monotonic on $(0, 1)$.*

The facts that (3) \implies (1) and (3) \implies (2) follow from the Schur product theorem [29]. However, the converse results are highly nontrivial.

In the present paper, we consider moments of measures on the line rather than Fourier coefficients, so power moments rather than complex exponential moments. Hence we study functions F mapping moment sequences entrywise into themselves, i.e., such that for every admissible measure μ , there exists an admissible measure $\sigma = \sigma_\mu$ satisfying

$$F(s_k(\mu)) = s_k(\sigma) \quad \forall k \geq 0.$$

Equivalently, by Theorem 2.2, we study entrywise endomorphisms of the cone of positive Hankel matrices with real entries. The next theorem, the first in a series to be demonstrated below, gives an idea of the type of positive Hankel-matrix preservers we seek.

Theorem 2.5. *A function $F : \mathbb{R} \rightarrow \mathbb{R}$ maps $\mathcal{M}([-1, 1])$ into itself when applied entrywise, if and only if F is the restriction to \mathbb{R} of an absolutely monotonic entire function.*

In particular, Theorem 2.5 strengthens Theorem 2.4, by relaxing the assumptions in [26, 28] to require positivity preservation only for Hankel matrices.

Our next result is a one-sided variant of the above characterizations, following Horn [17, Theorem 1.2]. Akin to Theorem 2.5, it arrives at the same conclusion under weaker assumptions than in [17].

Theorem 2.6. *A function $F : [0, \infty) \rightarrow \mathbb{R}$ maps $\mathcal{M}([0, 1])$ into itself when applied entrywise, if and only if F is absolutely monotonic on $(0, \infty)$, so non-decreasing, and $0 \leq F(0) \leq \lim_{\epsilon \rightarrow 0^+} F(\epsilon)$.*

In this paper we also study preservers of $\mathcal{M}([-1, 0])$, and show that these are classified as follows.

Theorem 2.7. *The following are equivalent for a function $F : \mathbb{R} \rightarrow \mathbb{R}$.*

- (1) *Applied entrywise, F maps $\mathcal{M}([-1, 0])$ into $\mathcal{M}((-\infty, 0])$.*
- (2) *There exists an absolutely monotonic entire function \tilde{F} such that*

$$F(x) = \begin{cases} \tilde{F}(x) & \text{if } x \in (0, \infty), \\ 0 & \text{if } x = 0, \\ -\tilde{F}(-x) & \text{if } x \in (-\infty, 0). \end{cases}$$

It is striking to observe the possibility of a discontinuity at the origin, in both of the previous theorems.

We will also derive a similar description of transformers $F[-] : \mathcal{M}([-1, 0]) \rightarrow \mathcal{M}([0, \infty))$; see Theorem 5.3. In this variant, we observe that F may also be discontinuous at 0.

The arguments used to show Theorem 2.4 and its one-sided variant by Schoenberg, Rudin, and Horn do not carry over to our setting involving positive Hankel matrices. This is due to the fact that the hypotheses in Theorems 2.5 and 2.6 are significantly weaker.

In the remainder of the paper, we will show how the assumptions in Theorems 2.5, 2.6, and 2.7 can be relaxed quite substantially. In doing so, our goal is to understand the minimal amount of information that is equivalent to the requirement that a function preserves $\mathcal{M}([0, 1])$ or $\mathcal{M}([-1, 1])$ when applied entrywise. We will demonstrate that requiring a function to preserve moments for measures supported on at most three points, is equivalent to preserving moments for all measures. In particular, this shows that preserving positivity for positive Hankel matrices of rank at most three implies positivity preservation for all positive matrices.

This latter point prompts a comparison to the case of Toeplitz matrices considered in [26]. Rudin proved that Theorem 2.4(3) holds if F preserves positivity on a two-parameter family of Toeplitz matrices with rank at most 3, namely

$$\{(a + b \cos((i - j)\theta))_{i,j \geq 1} : a, b \geq 0, a + b < 1\}, \quad (2.2)$$

where θ is a fixed real number such that θ/π is irrational. Similarly, the present work shows that for power moments, it suffices to work with families of positive Hankel matrices of rank at most three. Theorem 4.1(1) contains the precise details.

3. MOMENT TRANSFORMERS ON $[0, 1]$

Over the course of the next three sections, we will formulate and prove strengthened versions of the results stated in the Preliminaries.

Here, we provide two proofs of Theorem 2.6. The first is natural from the point of view of moments and Hankel matrices. The proof proceeds by first deriving from positivity considerations some inequalities satisfied by all moments transformers. We then obtain the desired characterization by appealing to classical results on completely monotonic functions. This is in the spirit of Lorch and Newman [21], who in turn are very much indebted to the original Hausdorff approach to the moment problem via summation rules and higher-order finite differences.

Recall that a function is said to be *completely monotonic* on an interval (a, b) if $x \mapsto f(-x)$ is absolutely monotonic on $(-b, -a)$, i.e., if $(-1)^k f^{(k)}(x) \geq 0$ for all $x \in (a, b)$. Similarly, a function is completely monotonic on an interval $I \subset \mathbb{R}$ if it is continuous on I and is completely monotonic on the interior of I .

Complete monotonicity can also be defined using finite differences. Let $\Delta_h^n f$ denote the n th forward difference of f with step size h :

$$\Delta_h^n f(x) := \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} f(x + kh).$$

Then f is completely monotonic on (a, b) if and only if $(-1)^k \Delta_h^n f(x) \geq 0$ for all non-negative integers n and for all x, h such that $a < x < x + h < \dots < x + nh < b$. See [32, Chapter IV] for more details on completely monotonic functions. Such functions were also characterized in a celebrated result of Bernstein.

Theorem 3.1 (Bernstein [32, Chapter IV, Theorem 12a]). *A function $f : [0, \infty) \rightarrow \mathbb{R}$ is completely monotonic on $0 \leq x < \infty$ if and only if*

$$f(x) = \int_0^\infty e^{-xt} d\mu(t)$$

for some positive measure μ .

Using the above results, we now provide our first proof of Theorem 2.6.

Proof 1 of Theorem 2.6. The ‘if’ part follows from two statements: (i) absolutely monotonic entire functions preserve positivity on all matrices of all orders, by the Schur product theorem; (ii) moment matrices from elements of $\mathcal{M}([0, 1])$ have zero entries if and only if $\mu = a\delta_0$ for some $a \geq 0$.

Conversely, suppose the function F preserves $\mathcal{M}([0, 1])$ when applied entrywise, i.e., given any $\mu \in \text{Meas}^+([0, 1])$, there exists $\sigma \in \text{Meas}^+([0, 1])$ such that

$$F(s_k(\mu)) = s_k(\sigma) \quad \forall k \geq 0.$$

Let $p(t) = a_0 t^0 + \dots + a_d t^d$ be a real polynomial such that $p(t) \geq 0$ on $[0, 1]$. Then,

$$0 \leq \int_0^1 p(t) d\sigma(t) = \sum_{k=0}^d a_k s_k(\sigma) = \sum_{k=0}^d a_k F(s_k(\mu)). \quad (3.1)$$

Now and below, we will employ (3.1) for a careful choice of measure μ and polynomial p , to deduce additional information about the function F . In the present situation, fix finitely many scalars c_j , $t_j > 0$ and an integer $n \geq 0$, and set

$$p(t) = (1 - t)^n \quad \text{and} \quad \mu = \sum_j e^{-t_j \alpha} c_j \delta_{e^{-t_j h}}, \quad (3.2)$$

where $\alpha > 0$ and $h > 0$. Now let $g(x) := \sum_j c_j e^{-t_j x}$, and apply (3.1) to see that the forward finite differences of $F \circ g$ alternate in sign. That is,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} F \left(\sum_j c_j e^{-t_j \alpha - t_j k h} \right) \geq 0,$$

so $(-1)^n \Delta_h^n (F \circ g)(\alpha) \geq 0$. As this holds for all α , $h > 0$ and all $n \geq 0$, it follows that $F \circ g : (0, \infty) \rightarrow (0, \infty)$ is completely monotonic for all μ as in (3.2). Using the weak density of such measures in $\text{Meas}^+((0, \infty))$, together with Bernstein's theorem (Theorem 3.1), it follows that $F \circ g$ is completely monotonic on $(0, \infty)$ for all completely monotonic functions $g : (0, \infty) \rightarrow (0, \infty)$. Finally, a theorem of Lorch and Newman [21, Theorem 5] now gives that $F : (0, \infty) \rightarrow (0, \infty)$ is absolutely monotonic. \square

Our second proof of Theorem 2.6 involves first strengthening the theorem, as mentioned in the introduction. We show that if F preserves positivity for 2×2 matrices, and sends $\mathcal{M}(\{1, u_0\})$ to $\mathcal{M}(\mathbb{R})$ for a single $u_0 \in (0, 1)$, then F is absolutely monotonic on $(0, \infty)$.

Theorem 3.2. *Fix $u_0 \in (0, 1)$. Given a function $F : [0, \infty) \rightarrow \mathbb{R}$, the following are equivalent.*

- (1) *Applied entrywise, F maps $\mathcal{M}(\{1, u_0\})$ into $\mathcal{M}(\mathbb{R})$, and $F(a)F(b) \geq F(\sqrt{ab})^2$ for all $a, b \geq 0$.*
- (2) *Applied entrywise, F maps $\mathcal{M}([0, 1])$ into itself.*
- (3) *The function F agrees on $(0, \infty)$ with an absolutely monotonic entire function and $0 \leq F(0) \leq \lim_{\epsilon \rightarrow 0^+} F(\epsilon)$.*

If F is known to be continuous on $(0, \infty)$, then the second hypothesis in (1) may be omitted.

Note that assertion (1) is significantly weaker than the requirement that F preserves $\mathcal{M}([0, 1])$. The rest of this subsection is devoted to proving Theorem 3.2; the proof requires results on functions preserving positivity for matrices of a fixed dimension.

Definition 3.3. Given a function $F : \mathbb{R} \rightarrow \mathbb{R}$, the function $F[-]$ acts on the set of real matrices by applying F entrywise:

$$F[A] := (F(a_{ij})) \quad \text{for any real matrix } A = (a_{ij}).$$

Given a subset $I \subset \mathbb{R}$, denote by $\mathcal{P}_N(I)$ the set of positive $N \times N$ matrices with entries in I . For $1 \leq k \leq N$, let $\mathcal{P}_N^k(I)$ denote the matrices in $\mathcal{P}_N(I)$ of rank at most k .

An observation on positivity preservers made by Löwner and developed by Horn [17] provides the following necessary condition for a function to preserve positivity on $\mathcal{P}_N((0, \infty))$ when applied entrywise.

Theorem 3.4 (Horn). *If a continuous function $F : (0, \infty) \rightarrow \mathbb{R}$ is such that $F[-] : \mathcal{P}_N((0, \infty)) \rightarrow \mathcal{P}_N(\mathbb{R})$, then $F \in C^{N-3}((0, \infty))$ and $F^{(k)}(x) \geq 0$ for all $x > 0$ and all $0 \leq k \leq N-3$. Moreover, if it is known that $F \in C^{N-1}((0, \infty))$, then $F^{(k)}(x) \geq 0$ for all $x > 0$ and all $0 \leq k \leq N-1$.*

As shown in [13, Theorem 4.1], the same result can be obtained by working only with a particular family of rank-two matrices, and without the continuity assumption.

In the next theorem, Horn's hypotheses are further relaxed by utilising only Hankel matrices.

Theorem 3.5. *Suppose $F : I \rightarrow \mathbb{R}$, where $I := (0, \rho)$ and $0 < \rho \leq \infty$. Fix $u_0 \in (0, 1)$ and an integer $N \geq 2$, and let $\mathbf{u} := (1, u_0, \dots, u_0^{N-1})^T$. Suppose $F[-]$ preserves positivity on $\mathcal{P}_2^1((0, \rho))$, and $F[A] \in \mathcal{P}_N(\mathbb{R})$ for the family of Hankel matrices*

$$\{A = a\mathbf{1}_{N \times N} + b\mathbf{u}\mathbf{u}^T : a \in [0, \rho), b \in [0, \rho - a), 0 < a + b < \rho\}. \quad (3.3)$$

Then $F \in C^{N-3}(I)$, with

$$F^{(k)}(x) \geq 0 \quad \forall x \in I, 0 \leq k \leq N-3,$$

and $F^{(N-3)}$ is a convex non-decreasing function on I . If, further, $F \in C^{N-1}(I)$, then $F^{(k)}(x) \geq 0$ for all $x \in I$ and $0 \leq k \leq N-1$.

Finally, if F is assumed to be continuous on $(0, \rho)$, then the assumption that F preserves positivity on $\mathcal{P}_2^1((0, \rho))$ is not necessary.

To prove Theorem 3.5, our first step is to identify all rank-one Hankel matrices. As this result is used throughout the paper, we isolate it for convenience.

Lemma 3.6. *A rank-one $N \times N$ matrix $\mathbf{u}\mathbf{u}^T$, with entries in any field, is Hankel if and only if either the successive entries of \mathbf{u} are in a geometric progression, or all entries but the last are 0. More precisely, the matrix $\mathbf{u}\mathbf{u}^T$ is Hankel if and only if*

$$u_j = \begin{cases} u_1(u_2/u_1)^{j-1} & \text{if } u_1 \neq 0, \\ 0 & \text{if } u_1 = 0 \text{ and } 1 \leq j < N. \end{cases} \quad (3.4)$$

Proof. Each principal 3×3 block submatrix of $\mathbf{u}\mathbf{u}^T$ with successive rows and columns is of the form

$$\begin{pmatrix} u_{j-1}^2 & u_{j-1}u_j & u_{j-1}u_{j+1} \\ u_ju_{j-1} & u_j^2 & u_ju_{j+1} \\ u_{j+1}u_{j-1} & u_{j+1}u_j & u_{j+1}^2 \end{pmatrix},$$

whence $u_{j-1}u_{j+1} = u_j^2$ for all $j \geq 2$. Identity (3.4) follows immediately. \square

Using the above result, we can now prove Theorem 3.5.

Proof of Theorem 3.5. If $F \in C(I)$, then the result follows by repeating the argument in [17, Theorem 1.2], but with the vector α replaced by a vector $\mathbf{u} \in \mathbb{R}^N$ as in Lemma 3.6.

Now suppose F is an arbitrary function, which is not identically zero on $(0, \rho)$; we claim that F must be continuous. We first show that $F(x) \neq 0$ for all $x \in (0, \rho)$. Indeed, suppose $F(c) = 0$ for some $c \in (0, \rho)$. Given $d \in (c, \rho)$, define a sufficiently long geometric progression $u'_0 = c, \dots, u'_n = d$, such that $u'_{n+1} \in (d, \rho)$. By considering the matrices

$$F[A_j], \quad \text{where } A_j := \begin{pmatrix} u'_j & u'_{j+1} \\ u'_{j+1} & u'_{j+2} \end{pmatrix}, \quad 0 \leq j \leq n-1,$$

we obtain that $F(d) = 0$ for all $d \in (c, \rho)$. A similar argument applies to $d \in (0, c)$, showing that $F \equiv 0$ on $(0, \rho)$.

Next, since $F[-]$ preserves positivity on $\mathcal{P}_2^1((0, \rho))$ and is positive on $(0, \rho)$, it follows that $g : x \mapsto \log F(e^x)$ is midpoint convex on the interval $(-\infty, \log \rho)$. Moreover, an argument similar to the above shows that on any interval $(a, b) \subset (-\infty, \log \rho)$ with

$b < \log \rho$, the function F is bounded above by $F(b)$. Hence, by [25, Theorem 71.C], the function g is necessarily continuous on $(-\infty, \log \rho)$, and so F is continuous on $(0, \rho)$. This proves the result in the general case. \square

Finally, we turn to the proof of Theorem 3.2, which provides a second proof of Theorem 2.6 which is more informative. We first observe that Theorem 3.5 can be reformulated in terms of moment sequences, using the fact that the matrices occurring in the statement of the theorem can be realized as truncations of positive Hankel matrices. To that end, we introduce the following notation.

Definition 3.7. Given $k \geq 0$ and $I \subset \mathbb{R}$, let the corresponding *truncated moment sequences* be the elements of the set

$$\mathcal{M}_k(I) := \{(s_0(\mu), \dots, s_k(\mu)) : \mu \in \text{Meas}(I)^+\}. \quad (3.5)$$

In view of Hamburger's Theorem for such sequences (see [1, Theorem 2.6.3]), a Hankel matrix with entries in the first and last columns given by

$$s_0, \dots, s_{N-1} \quad \text{and} \quad s_{N-1}, \dots, s_{2N-2}$$

is positive if and only if $(s_0, \dots, s_{2N-2}) \in \mathcal{M}_{2N-2}(\mathbb{R})$. Furthermore, in order to show continuity in Theorem 3.5 we only required 2×2 submatrices. Thus, in the language of truncated moment sequences, Theorem 3.5 for $\rho = \infty$ is equivalent to the following.

Theorem 3.8. Suppose $F : (0, \infty) \rightarrow \mathbb{R}$. Fix $N \geq 2$, and suppose $F[-]$ maps $\mathcal{M}_{2N-2}(\{1, u_0\})$ to $\mathcal{M}_{2N-2}(\mathbb{R})$ for some $u_0 \in (0, 1)$, and $\mathcal{M}_2(\{u\})$ to $\mathcal{M}_2(\mathbb{R})$ for all $u \in (0, 1)$. Then $F \in C^{N-3}((0, \infty))$, with

$$F^{(k)}(x) \geq 0 \quad \forall x > 0, \quad 0 \leq k \leq N-3,$$

and $F^{(N-3)}$ is a convex non-decreasing function on $(0, \infty)$. If, further, it is known that $F \in C^{N-1}((0, \infty))$, then $F^{(k)}(x) \geq 0$ for all $x > 0$ and $0 \leq k \leq N-1$.

If F is continuous on $(0, \infty)$, then the assumption that $F[\mathcal{M}_2(\{u\})] \subset \mathcal{M}_2(\mathbb{R})$ for all $u \in (0, 1)$ may be omitted.

Truncated moment sequences may be used to reformulate assertion (1) in Theorem 3.2 as follows: $F[-]$ maps $\mathcal{M}(\{1, u_0\})$ into $\mathcal{M}(\mathbb{R})$ and $\mathcal{M}_2(\{a\})$ into $\mathcal{M}_2(\mathbb{R})$ for all $a \in [0, 1]$.

We now turn to the proof of Theorem 3.2, with the help of Theorem 3.8. We will appeal to the following classical result, which will also be useful later.

Theorem 3.9 ([32, Chapter IV, Theorem 3a]). *If f is absolutely monotonic on $[a, b]$, then it can be extended analytically into the complex plane, with its extension analytic in the disc of radius $b - a$ centered at a .*

Proof of Theorem 3.2. Clearly (2) \implies (1). Next, assume (3) holds, and suppose $\mu \in \text{Meas}^+([0, 1])$. If $\mu = a\delta_0$ for some $a \geq 0$ then (2) is immediate; henceforth we will assume H_μ has no zero entries. Now, $F[H_\mu]$ is positive for $\mu \in \text{Meas}^+([0, 1])$, by the Schur product theorem and the fact that the only moment matrices arising from elements of $\mathcal{M}([0, 1])$ with zero entries come from $\mathcal{M}(\{0\})$. Clearly $F[s(\mu)]$ is uniformly bounded, hence comes from a unique measure σ supported on $[-1, 1]$, by Theorem 2.2.

Recalling Definition 2.3, we have that

$$F[H_\mu]^{(1)} = \sum_{n \geq 0} c_n [H_\mu^{(1)}]^{on},$$

where $F(x) = \sum_{n \geq 0} c_n x^n$ by the hypotheses and Theorem 3.9. Note that $F[H_\mu]^{(1)}$ is positive, by the above computation and Theorem 2.2, since μ is supported on $[0, 1]$. By the same result, $\sigma \in \text{Meas}^+([0, 1])$, which gives (2).

It remains to show (1) \implies (3). By Theorem 3.8, it holds that $F^{(k)}(x) \geq 0$ for all $x > 0$ and all $k \geq 0$. Theorem 3.9 now gives the result, apart from the assertion about $F(0)$, but this is immediate. \square

3.1. Hankel-matrix positivity preservers in fixed dimension. We conclude this section by addressing briefly the fixed-dimension case for powers and analytic functions, as studied by FitzGerald and Horn, and also in previous work by the authors [3, 9, 12]. Our first result shows that considerations of Hankel matrices may be used to strengthen the main result in [3].

Theorem 3.10. *Fix $\rho > 0$ and integers $N \geq 1$ and $M \geq 0$, and let $F(z) = \sum_{j=0}^{N-1} c_j z^j + c' z^M$ be a polynomial with real coefficients. The following are equivalent.*

- (1) $F[-]$ preserves positivity on $\mathcal{P}_N(\overline{D}(0, \rho))$, where $\overline{D}(0, \rho)$ is the closed disc in the complex plane with center 0 and radius ρ .
- (2) The coefficients c_j satisfy either $c_0, \dots, c_{N-1}, c' \geq 0$, or $c_0, \dots, c_{N-1} > 0$ and $c' \geq -\mathcal{C}(\mathbf{c}; z^M; N, \rho)^{-1}$, where

$$\mathcal{C}(\mathbf{c}; z^M; N, \rho) := \sum_{j=0}^{N-1} \binom{M}{j}^2 \binom{M-j-1}{N-j-1}^2 \frac{\rho^{M-j}}{c_j}.$$

- (3) $F[-]$ preserves positivity on Hankel matrices in $\mathcal{P}_N^1((0, \rho))$.

Remark 3.11. As we show below, assumption (3) in Theorem 3.10 can be relaxed further, by assuming F preserves positivity on a distinguished family of Hankel matrices. More precisely, it can be replaced by

- (3') $F[-]$ preserves positivity on two sequences of rank-one Hankel matrices,

$$\{b^n \rho \mathbf{u}(b) \mathbf{u}(b)^T, \rho \mathbf{u}(b^n) \mathbf{u}(b^n)^T : n \geq 1\}, \quad \text{for any fixed } b \in (0, 1),$$

where

$$\mathbf{u}(\epsilon) := (1 - \epsilon, (1 - \epsilon)^2, \dots, (1 - \epsilon)^N)^T, \quad \text{for any } \epsilon \in (0, 1). \quad (3.6)$$

Note that $\mathbf{u}(\epsilon) \mathbf{u}(\epsilon)^T \in \mathcal{P}_N^1(\mathbb{R})$ is Hankel, by Lemma 3.6.

Theorem 3.10 and Remark 3.11 strengthen [3, Theorem 1.1] by weakening its hypothesis (3), which previously required positivity preservation on all of $\mathcal{P}_N^1((0, \rho))$. It is striking to compare this N -dimensional parameter space to the current version (3'), which uses a countable subset of Hankel matrices. If $N > 1$, this is indeed minimal information required to derive Theorem 3.10(2), since the extreme critical value $\mathcal{C}(\mathbf{c}; z^M; N, \rho)$ cannot be attained on any finite set of matrices in $\mathcal{P}_N^1((0, \rho))$.

As a first step towards the proof of Theorem 3.10, we recall from [3, Lemma 2.4] that, under suitable differentiability assumptions, the conclusions of Theorem 3.5 still

hold if one considers only rank-one matrices. We now formulate a slightly stronger version of this result.

Proposition 3.12. *Let $F \in C^\infty((-\rho, \rho))$, where $0 < \rho \leq \infty$. Fix a vector $\mathbf{u} \in (0, \sqrt{\rho})^N$ with distinct coordinates, and suppose $F[b\mathbf{u}\mathbf{u}^T] \in \mathcal{P}_N(\mathbb{R})$ for all $b \in (0, 1)$. Then the first N non-zero derivatives of F at 0 are strictly positive.*

Proof. For ease of exposition, we will assume F has at least N non-zero derivatives at 0, say of orders $m_1 < \dots < m_N$, where $m_1 \geq 0$. By results on generalized Vandermonde determinants [11, Chapter XIII, §8, Example 1], the vectors $\{\mathbf{u}^{\circ m_j} : 1 \leq j \leq N\}$ are linearly independent. Now, by Taylor's theorem,

$$F[b\mathbf{u}\mathbf{u}^T] = \sum_{j=1}^N \frac{F^{(m_j)}(0)}{m_j!} b^{m_j} \mathbf{u}^{\circ m_j} (\mathbf{u}^{\circ m_j})^T + o(b^{m_N}). \quad (3.7)$$

For each $1 \leq k \leq N$, choose $\mathbf{v}_k \in \mathbb{R}^N$ such that $\mathbf{v}_k^T \mathbf{u}^{\circ m_j} = \delta_{j,k} m_j!$. Then,

$$b^{-m_k} \mathbf{v}_k^T F[b\mathbf{u}\mathbf{u}^T] \mathbf{v}_k = F^{(m_k)}(0) + o(b^{m_N - m_k}) \geq 0,$$

and letting $b \rightarrow 0^+$ concludes the proof. \square

Remark 3.13. As is clear from the proof, in Proposition 3.12 it suffices to work with a positive real sequence (b_n) which converges to 0, rather than with all $b \in (0, 1)$.

We now use Proposition 3.12 to prove the theorem.

Proof of Theorem 3.10. In view of Remark 3.11 and [3, Theorem 1.1], it suffices to show that (3') \implies (2).

Assume (3') holds, and consider first the sequence $b^n \rho \mathbf{u}(b) \mathbf{u}(b)^T$. If $0 \leq M < N$, then the result follows from Proposition 3.12 via Remark 3.13, since the critical value is precisely $\mathcal{C}(\mathbf{c}; z^M; N, \rho) = c_M^{-1}$. Now suppose $M \geq N$. Again using Remark 3.13 and Proposition 3.12, either c_0, \dots, c_{N-1} and c' are all non-negative, or else we have that $c_0, \dots, c_{N-1} > 0 > c_M$.

In the latter case, to prove that $c_M \geq -\mathcal{C}(\mathbf{c}; z^M; N, \rho)^{-1}$, we use the sequence $\rho \mathbf{u}(b^n) \mathbf{u}(b^n)^T$, where $\mathbf{u}(b^n)$ is defined as in (3.6). Let $\mathbf{u}_n := \sqrt{\rho} \mathbf{u}(b^n)$ for $n \geq 1$. Then [3, Equation (3.11)] implies that $0 \leq \det |c_M|^{-1} F[\mathbf{u}_n \mathbf{u}_n^T]$, and so

$$|c_M|^{-1} \geq \sum_{j=0}^{N-1} \frac{s_{\mu(M, N, j)}(\mathbf{u}_n)^2}{c_j},$$

where $\mu(M, N, j)$ is the hook partition $(M - N + 1, 1, \dots, 1, 0, \dots, 0)$, with $N - j - 1$ ones after the first entry and then j zeros, and $s_{\mu(M, N, j)}$ is the corresponding Schur polynomial. As $n \rightarrow \infty$, so $\mathbf{u}_n \rightarrow \sqrt{\rho}(1, \dots, 1)^T$. The Weyl Character Formula in type A gives that $s_{\mu(M, N, j)}(1, \dots, 1) = \binom{M}{j} \binom{M - j - 1}{N - j - 1}$, and it follows that

$$|c_M|^{-1} \geq \sum_{j=0}^{N-1} \binom{M}{j}^2 \binom{M - j - 1}{N - j - 1}^2 \frac{\rho^{M-j}}{c_j} = \mathcal{C}(\mathbf{c}; z^M; N, \rho).$$

Thus (2) holds, and this concludes the proof. \square

Finally, we consider the question of which real powers preserve positivity on $N \times N$ Hankel matrices. Recall that the Schur product theorem guarantees that integer powers $x \mapsto x^k$ preserve positivity on $\mathcal{P}_N((0, \infty))$. It is natural to ask if any other real powers do so. In [9], FitzGerald and Horn solved this problem, and uncovered an intriguing transition. In their main result, they show that the power function $x \mapsto x^\alpha$ preserves positivity entrywise on $\mathcal{P}_N((0, \infty))$ if and only if α is a non-negative integer or $\alpha \geq N - 2$. The value $N - 2$ is known in the literature as the *critical exponent* for preserving positivity.

As shown in [12], the critical exponent remains unchanged upon restricting the problem to preserving positivity on $\mathcal{P}_N^k((0, \infty))$ for any $k \geq 2$. More precisely, for each non-integral $\alpha \in (0, N - 2)$, there exists a rank-two matrix $A \in \mathcal{P}_N^2((0, \infty))$ such that $A^{\circ\alpha} \notin \mathcal{P}_N$; see [12] for more details.

As we now show, the result does not change when restricted to the set of positive semidefinite Hankel matrices.

Proposition 3.14. *Let $2 \leq k \leq N$ and let $\alpha \in \mathbb{R}$. The following are equivalent.*

- (1) *The power function $x \mapsto x^\alpha$ preserves positivity when applied entrywise to Hankel matrices in $\mathcal{P}_N^k((0, \infty))$.*
- (2) *The power α is a non-negative integer or $\alpha \geq N - 2$.*

Moreover, there exists a Hankel matrix $A \in \mathcal{P}_N^2((0, \infty))$ such that $A^{\circ\alpha} \notin \mathcal{P}_N$ for all non-integral $\alpha \in (0, N - 2)$.

Proof. By the main result in [19], for pairwise distinct real numbers $x_1, \dots, x_N > 0$, the matrix $((1 + x_i x_j)^\alpha)_{i,j=1}^N$ is positive semidefinite if and only if α is a non-negative integer or $\alpha \geq N - 2$. The result now follows immediately, by Lemma 3.6. \square

Remark 3.15. We conclude by explaining why Theorem 3.2 provides a minimal set of rank-constrained positive semidefinite matrices for which positivity preservation is equivalent to absolute monotonicity. A smaller set of rank-constrained matrices could not include a sequence of matrices in $\mathcal{P}_N^2([0, \infty))$ of unbounded dimension, hence would be contained in $P'_N := \bigcup_{n=1}^N \mathcal{P}_n^2([0, \infty)) \cup \bigcup_{n \in \mathbb{N}} \mathcal{P}_n^1([0, \infty))$ for some $N \geq 1$. However, as noted above, the map $x \mapsto x^\alpha$ preserves positivity on P'_N for all $\alpha \geq N - 2$, and such a function may be non-analytic.

4. MOMENT TRANSFORMERS ON $[-1, 1]$

Equipped with the one-sided result from Theorem 3.2, we now classify the functions which map the set $\mathcal{M}([-1, 1])$ into $\mathcal{M}(\mathbb{R})$ when applied entrywise. The goal of this section is to prove the following strengthening of Theorem 2.5, in the spirit of Theorem 3.2.

Theorem 4.1. *The following are equivalent for a function $F : \mathbb{R} \rightarrow \mathbb{R}$.*

- (1) *$F[-]$ maps the sequences $\bigcup_{u \in (0,1)} \mathcal{M}(\{-1, u, 1\})$ into $\mathcal{M}(\mathbb{R})$.*
- (2) *$F[-]$ maps $\mathcal{M}([-1, 1])$ into $\mathcal{M}(\mathbb{R})$.*
- (3) *F is the restriction to \mathbb{R} of an absolutely monotonic entire function.*

The proof of Theorem 4.1 requires new ideas, as previous techniques to prove analogous results are not amenable to the weaker Hankel setting; see Remark 4.6.

Recall the notion of truncated moment sequences from Definition 3.7.

Lemma 4.2. *If $F : \mathbb{R} \rightarrow \mathbb{R}$ maps entrywise the sequences*

$$\mathcal{M}_2(\{-1\}) \cup \bigcup_{u \in (0,1)} \mathcal{M}_2(\{u\}) \quad (4.1)$$

into $\mathcal{M}_2(\mathbb{R})$, then F is locally bounded. If F is known to be continuous on $(0, \infty)$, then the set (4.1) may be replaced by $\mathcal{M}_2(\{-1\})$.

Proof. As in the proof of Theorem 3.5, the assumption implies that F is continuous, whence locally bounded, on $(0, \infty)$. Now let $\mu = a\delta_{-1}$ for any $a > 0$. By considering the leading principal 2×2 submatrix of $F[H_\mu]$, where H_μ is the Hankel matrix associated to the measure μ , it follows that $|F(-a)| \leq F(a)$. \square

Remark 4.3. In light of Lemma 4.2 and Theorem 2.2, assertion (1) in Theorem 4.1 can be strengthened to

$$F[-] : \bigcup_{u \in (0,1)} \mathcal{M}(\{-1, u, 1\}) \rightarrow \mathcal{M}([-1, 1]). \quad (4.2)$$

The next step is to use (4.2) to establish the continuity of F on \mathbb{R} .

Proposition 4.4. *Fix $v_0 \in (0, 1)$, and suppose $F : \mathbb{R} \rightarrow \mathbb{R}$ maps entrywise*

$$\mathcal{M}_3(\{-1, v_0\}) \cup \bigcup_{u \in (0,1)} \mathcal{M}_4(\{1, u\})$$

into $\mathcal{M}_3([-1, 1]) \cup \mathcal{M}_4([-1, 1])$. Then F is continuous on \mathbb{R} .

Proof. By Theorem 3.8 for $N = 3$, and our assumptions, F is continuous, positive, and non-decreasing on $(0, \infty)$. In particular, F has a right-hand limit at 0, and

$$0 \leq F(0) \leq \lim_{\epsilon \rightarrow 0^+} F(\epsilon). \quad (4.3)$$

We now fix $v_0 \in (0, 1)$ and use the truncated moment sequences in $\mathcal{M}_3(\{-1, v_0\})$ to prove two-sided continuity of F at all points in $(-\infty, 0]$. Fix $\beta \geq 0$, and for $b > 0$, let

$$a := \beta + bv_0, \quad \text{and} \quad \mu = a\delta_{-1} + b\delta_{v_0}.$$

By assumption, we have that $F[-] : \mathcal{M}_3(\{-1, v_0\}) \rightarrow \mathcal{M}_3([-1, 1])$, so there exists $\sigma \in \text{Meas}^+[-1, 1]$ such that

$$(F(s_0(\mu)), \dots, F(s_3(\mu))) = (s_0(\sigma), \dots, s_3(\sigma)).$$

If the polynomials $p_\pm(t) := (1 \pm t)(1 - t^2)$ then,

$$\int_{-1}^1 p_\pm(t) d\sigma \geq 0,$$

since $p_\pm(t)$ are non-negative on $[-1, 1]$. Hence (3.1) gives that

$$F(a+b) - F(a+bv_0^2) \geq \pm (F(-a+bv_0) - F(-a+bv_0^3)),$$

or, equivalently,

$$F(\beta + b + bv_0) - F(\beta + bv_0 + bv_0^2) \geq |F(-\beta) - F(-\beta - b(v_0 - v_0^3))|.$$

Letting $b \rightarrow 0^+$ and using the continuity of F on $(0, \infty)$, we conclude that F is left continuous at $-\beta$. We proceed similarly to show right continuity of F at $-\beta$; now let

$$a := \beta + bv_0^3 \quad \text{and} \quad \mu = a\delta_{-1} + b\delta_{v_0},$$

where again $b > 0$. Using the same argument as above, we see that

$$F(\beta + b + bv_0^3) - F(\beta + bv_0^2 + bv_0^3) \geq |F(-\beta + b(v_0 - v_0^3)) - F(-\beta)|,$$

and the right continuity follows on letting $b \rightarrow 0^+$. \square

With the continuity in hand, we can now complete the proof of Theorem 4.1.

Proof of Theorem 4.1. Clearly (3) \implies (2) \implies (1). Now suppose (1) holds. By Proposition 4.4, F is continuous. Using a classical mollifier argument, we claim that it suffices to prove (3) for F smooth. Indeed, for any $\delta > 0$, choose $g_\delta \in C^\infty(\mathbb{R})$ such that g_δ is non-negative, is supported on $(0, \delta)$, and integrates to 1, and let

$$F_\delta(x) := \int_0^\delta F(x+t)g_\delta(t) dt \quad \forall x \in \mathbb{R}.$$

As the function $x \mapsto F(x+t)$ satisfies hypothesis (1) of the theorem, so does the smooth function F_δ . Moreover, since

$$|F(x) - F_\delta(x)| = \left| \int_0^\delta [F(x) - F(x+t)]g_\delta(t) dt \right| \leq \sup_{0 \leq t \leq \delta} |F(x) - F(x+t)| \quad \forall x \in \mathbb{R},$$

it follows that F_δ converges locally uniformly to F as $\delta \rightarrow 0^+$. We therefore assume henceforth that F is smooth.

Since $F[-] : \mathcal{M}(\{-1, \sqrt{u_0}\}) \rightarrow \mathcal{M}(\mathbb{R})$ for any $u_0 \in (0, 1)$, it follows from Theorem 3.2 that F agrees on $(0, \infty)$ with an absolutely monotonic entire function \tilde{F} . Now let $\mu := a\delta_{-1} + e^x\delta_{e^{-h}}$, where $a > 0$, $h > 0$ and $x \in \mathbb{R}$, and let $p_{\pm,n}(t) := (1 \pm t)(1 - t^2)^n$. Then $p_{\pm,n}(t)$ is non-negative for all $t \in [-1, 1]$ and all $n \geq 0$. Applying (3.1) gives that

$$\left| \sum_{k=0}^n \binom{n}{k} (-1)^k F(a + e^{x-2kh}) \right| \geq \left| \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} F(-a + e^{x-(2k+1)h}) \right|. \quad (4.4)$$

Let $H_{\pm,a}(x) := F(\pm a + e^x)$; dividing (4.4) by h^n and taking $h \rightarrow 0^+$, we see that

$$|H_{+,a}^{(n)}(x)| \geq |H_{-,a}^{(n)}(x)|.$$

Since $H_{+,a}$ is real analytic, we conclude that the Taylor series for $H_{-,a}$ has a positive radius of convergence everywhere, so $H_{-,a}$ is real analytic on \mathbb{R} . The change of variable $x = \log(y + a)$ has a convergent power-series expansion for $|y| < a$. It follows that $y \mapsto F(y)$ is real analytic on \mathbb{R} , hence is the restriction of \tilde{F} . \square

Remark 4.5. It follows from the proof of Theorem 4.1 that if F is assumed to be infinitely differentiable, then the set $\bigcup_{u \in (0,1)} \mathcal{M}(\{-1, u, 1\})$ in hypothesis (1) may be replaced with $\bigcup_{u \in (0,1)} \mathcal{M}(\{-1, u\})$. Indeed, Lemma 4.2 and Proposition 4.4 do not require the point 1 to belong to the support of all sets in this hypothesis. It is when we use a mollifier argument to move from continuous to smooth functions that the stronger assumption is needed.

Remark 4.6. Recall that Rudin [26] showed that F must be analytic on \mathbb{R} and absolutely monotonic on $(0, 1)$ if $F[-]$ preserves positivity for the two-parameter family of Toeplitz matrices defined in 2.2. A natural strategy to prove Theorem 4.1 would be to show that there exists $\theta \in \mathbb{R}$ with θ/π irrational, such that the matrices $(\cos((i-j)\theta))_{i,j=1}^n$ can be embedded into positive Hankel matrices, for all sufficiently

large n . However, this is not possible: given $0 < m_1 < m_2$ such that $\cos(m_1\theta) < 0$ and $\cos(m_2\theta) < 0$, if there were a measure $\mu \in \text{Meas}^+([-1, 1])$ such that $\cos(m_j\theta) = s_{k_j}(\mu)$ for $j = 1$ and $j = 2$, then, by the Toeplitz property, k_1 , k_2 , and $k_1 + k_2$ must all be odd, which is impossible.

5. MOMENT TRANSFORMERS ON $[-1, 0]$

We now study the structure of endomorphisms of $\mathcal{M}([-1, 0])$. The following characterization result reveals that such functions may be discontinuous at the origin. This is in contrast to Theorem 4.1.

Theorem 5.1. *Suppose $F : \mathbb{R} \rightarrow \mathbb{R}$ and $u_0 \in (0, 1)$. The following are equivalent.*

- (1) $F[-]$ maps entrywise the sequences

$$\mathcal{M}_2(\{0\}) \cup \mathcal{M}(\{-1, -u_0\}) \cup \bigcup_{u \in (0, 1)} \mathcal{M}_4(\{-u\})$$

into $\mathcal{M}_2((-\infty, 0]) \cup \mathcal{M}((-\infty, 0]) \cup \mathcal{M}_4((-\infty, 0])$.

- (2) $F[-]$ preserves $\mathcal{M}([-1, 0])$.

- (3) There exists an absolutely monotonic entire function \tilde{F} such that

$$F(x) = \begin{cases} \tilde{F}(x) & \text{if } x \in (0, \infty), \\ 0 & \text{if } x = 0, \\ -\tilde{F}(-x) & \text{if } x \in (-\infty, 0). \end{cases}$$

In particular, the function F is odd, but may be discontinuous at 0.

Proof. To show that (3) \implies (2), note first that if $\mu \in \text{Meas}^+([-1, 0])$, so that $\mu = a\delta_0$ for some a , then $F[H_\mu] = H_{F(a)\delta_0}$, so we may assume μ is not of this form, whence the Hankel matrix H_μ has no zero entries, and the moment sequence alternates in sign and is uniformly bounded, by Theorem 2.2. In particular,

$$F(s_{2k}(\mu)) = \tilde{F}(s_{2k}(\mu)) \quad \text{and} \quad F(s_{2k+1}(\mu)) = -\tilde{F}(-s_{2k+1}(\mu)) \quad \forall k \geq 0.$$

Recalling the form of the Hankel matrix $H_{\delta_{-1}}$, it follows that

$$F[H_\mu] = H_{\delta_{-1}} \circ \tilde{F}[H_{\delta_{-1}} \circ H_\mu] \tag{5.1}$$

where \circ denotes the entrywise matrix product. This shows (2) because $F[-]$ is the composite of two operations: $\tilde{F}[-]$, which preserves $\mathcal{M}([0, 1])$ by Theorem 3.2, and entrywise multiplication by the matrix $H_{\delta_{-1}}$, which maps H_μ for some measure μ to the Hankel matrix of the reflection of μ about the origin.

That (2) \implies (1) is immediate. We now prove (1) \implies (3). Suppose (1) holds. Since

$$F[H_{a\delta_0}] = (F(a) - F(0))H_{\delta_0} + F(0)H_{\delta_1} = H_{(F(a)-F(0))\delta_0 + F(0)\delta_1},$$

the uniqueness in Theorem 2.2 gives that $F(0) = 0$.

By considering only even rows and columns of Hankel matrices corresponding to moments in $\mathcal{M}_4(\{-u\})$ and $\mathcal{M}(\{-1, -u_0\})$, we have embeddings

$$\mathcal{M}_2(\{u^2\}) \hookrightarrow \mathcal{M}_4(\{-u\}) \quad \text{and} \quad \mathcal{M}(\{1, u_0^2\}) \hookrightarrow \mathcal{M}(\{-1, -u_0\}).$$

Thus $F[-]$ maps $\mathcal{M}_2(\{u^2\})$ into $\mathcal{M}_2(\mathbb{R})$, and $\mathcal{M}(\{1, u_0^2\})$ into $\mathcal{M}(\mathbb{R})$. Theorem 3.8 now gives that F agrees with an absolutely monotonic entire function \tilde{F} on $(0, \infty)$.

Next, considering $\mathcal{M}_2(\{-1\})$ gives that $|F(-a)| \leq F(a)$ for $a > 0$, whence F is locally bounded. In particular, F maps $\mathcal{M}(\{-1\})$ into $\mathcal{M}([-1, 0])$, by Theorem 2.2.

We conclude by showing that F is odd. Let $\mu = a\delta_{-1}$ for some $a > 0$ and note that $p_n(t) = (-t)^n(1+t)$ is non-negative on $[-1, 0]$ for any non-negative integer n . If $F[s(\mu)] = s(\sigma)$, then, by applying (3.1),

$$\begin{aligned} 0 &\leq \int_{-1}^0 p_n(t) d\sigma = (-1)^n (F(s_n(a\delta_{-1})) + F(s_{n+1}(a\delta_{-1}))) \\ &= (-1)^n (F((-1)^n a) + F((-1)^{n+1} a)). \end{aligned}$$

Taking $n = 0$ and 1 gives that $0 \leq F(a) + F(-a) \leq 0$, and the final claim follows. \square

Theorem 5.1 has the following consequence.

Corollary 5.2. *Define a checkerboard matrix to be any real matrix $A = (a_{ij})$ such that $(-1)^{i+j}a_{ij} > 0$ for all i, j . Given a function $F : \mathbb{R} \rightarrow \mathbb{R}$, the following are equivalent.*

- (1) *Applied entrywise, F maps the set of positive Hankel checkerboard matrices of all dimensions into itself.*
- (2) *Applied entrywise, F maps the set of positive checkerboard matrices of all dimensions into itself.*
- (3) *F is odd and agrees on $(0, \infty)$ with an absolutely monotonic entire function.*

We conclude this section with an even analogue of Theorem 5.1.

Theorem 5.3. *Let the function $F : \mathbb{R} \rightarrow \mathbb{R}$ and suppose $u_0 \in (0, 1)$. The following are equivalent.*

- (1) *$F[-]$ maps entrywise the sequences*

$$\mathcal{M}_2(\{0\}) \cup \mathcal{M}(\{-1, -u_0\}) \cup \bigcup_{u \in (0, 1)} \mathcal{M}_4(\{-u\})$$

into $\mathcal{M}_2([0, \infty)) \cup \mathcal{M}([0, \infty)) \cup \mathcal{M}_4([0, \infty))$.

- (2) *$F[-]$ sends $\mathcal{M}([-1, 0])$ to $\mathcal{M}([0, 1])$.*
- (3) *There exists an absolutely monotonic entire function \tilde{F} such that*

$$F(x) = \begin{cases} \tilde{F}(x) & \text{if } x \in (0, \infty), \\ \tilde{F}(-x) & \text{if } x \in (-\infty, 0). \end{cases}$$

Moreover, $0 \leq F(0) \leq \lim_{\epsilon \rightarrow 0} F(\epsilon)$.

Proof. This is similar to the proof of Theorem 5.1; to show that (3) \implies (1), one may use the polynomials $p_n(t) = t^n(1-t)$. We omit further details. \square

6. TRANSFORMERS WITH RESTRICTED DOMAIN

Schoenberg and Rudin's result, Theorem 2.4, characterizes positivity preservers for matrices with entries in $(-1, 1)$. With this in mind, we now show how results in the previous sections can be refined to apply similarly, with moments contained in a bounded interval.

For measures supported on $[-1, 1]$, the mass $s_0(\mu)$ dominates $|s_k(\mu)|$ for all $k \geq 0$. Studying positivity preservers for Hankel matrices with entries in a bounded interval $(-\rho, \rho)$ is therefore equivalent to working with measures such that $s_0(\mu) < \rho$.

Theorem 6.1. *Suppose $\rho \in (0, \infty]$, and let F be a function with domain $[0, \rho)$ or $(-\rho, \rho)$. The results in Sections 3, 4, and 5 still hold, once one restricts all moment sequences on which $F[-]$ acts to those with mass $s_0(\mu) < \rho$.*

Theorem 6.1 may be established by inspecting the proofs of the results in Sections 3, 4, and 5, with minimal modifications to ensure the arguments of F remain in the required domain.

As a consequence of Theorem 6.1, we obtain the following generalization of Schoenberg and Rudin's result. The domain is replaced by an arbitrary symmetric interval, and F is required to preserve positivity only on Hankel matrices.

Corollary 6.2. *Theorem 2.4 holds with $(-1, 1)$ and $(0, 1)$ replaced by $(-\rho, \rho)$ and $(0, \rho)$, respectively, for any $\rho \in (0, \infty]$. Furthermore, its hypotheses are equivalent to the assertion that $F[-]$ preserves positivity on Hankel matrices arising from moment sequences, with entries in $(-\rho, \rho)$ and rank at most 3.*

When the domain of F is a closed interval I , the situation is more complex; absolute monotonicity, or even continuity of F , does not extend automatically from the interior of I to its end points. This was already observed by Rudin via specific counterexamples; see Remark (a) at the end of [26]. To the best of our knowledge, characterization results in this setting are not known.

We now take a closer look at this phenomenon. We begin by characterizing those functions preserving positivity of Hankel matrices in $\mathcal{P}_N(I)$ for all N , where $I = [0, \rho]$ and $\rho \in (0, \infty)$.

Proposition 6.3. *Suppose $\rho \in (0, \infty)$ and $F : I \rightarrow \mathbb{R}$, where $I = [0, \rho]$. The following are equivalent.*

- (1) $F[-]$ preserves positivity on all positive Hankel matrices with entries in I .
- (2) F is absolutely monotonic on $[0, \rho)$ and $F(\rho) \geq \lim_{x \rightarrow \rho^-} F(x)$.
- (3) $F[-]$ preserves positivity on all positive matrices with entries in I .

If $I = [0, \rho]$ then the same equivalences hold, with (2) replaced by the requirement that F is absolutely monotonic on $[0, \rho)$.

Note the contrast with Theorem 3.2: if $F[-]$ is required only to preserve positive Hankel matrices arising from moment sequences, then F may be discontinuous at 0, but this cannot occur here.

Proof. Clearly (3) \implies (1). Next, suppose (1) holds and note that F is absolutely monotonic on $(0, \rho)$, by Theorem 6.1. Consider the positive Hankel matrices

$$H_a := \begin{pmatrix} a & 0 & a \\ 0 & a & a \\ a & a & 2a \end{pmatrix}, \quad \text{where } a \in [0, \rho/2).$$

As $F[H_a]$ is positive, so $0 \leq F(0) \leq F^+(0) := \lim_{a \rightarrow 0^+} F(a)$. Furthermore,

$$0 \leq \lim_{a \rightarrow 0^+} \det F[H_a] = -F^+(0)(F(0) - F^+(0))^2,$$

whence $F(0) = F^+(0)$, and F is right continuous at the origin. Finally, considering the first two leading principal minors of the Hankel matrix for the measure $(\rho - a)\delta_1 + a\delta_0$, where $a \rightarrow \rho^-$, gives that $F(\rho) \geq \lim_{a \rightarrow \rho^-} F(a)$. Hence (1) \implies (2).

Finally, suppose (2) holds. We first claim that if $A \in \mathcal{P}_N((-\infty, \rho])$ then the entries of A equalling ρ form a (possibly empty) block diagonal submatrix, upon suitably relabelling the indices. Indeed,

$$0 \leq \det \begin{pmatrix} \rho & \rho & a \\ \rho & \rho & \rho \\ a & \rho & \rho \end{pmatrix} = -\rho(\rho - a)^2 \implies a = \rho. \quad (6.1)$$

Now let B_A be the block-diagonal matrix with (i, j) th entry equal to 1 if $a_{ij} = \rho$ and 0 otherwise. If g is the continuous extension of $F|_{[0, \rho]}$ to ρ , then

$$F[A] = g[A] + (F(\rho) - g(\rho))B_A \geq 0,$$

since both matrices are positive semidefinite. Hence (2) \implies (3).

Finally, when $I = [0, \rho]$, that (2) \implies (3) \implies (1) is immediate, and (1) \implies (2) is shown as above. \square

Remark 6.4. A similar argument to Proposition 6.3 reveals that $F[-]$ preserves positivity on the set $\{\mathbf{s}(\mu) \in \mathcal{M}([0, 1]) : s_0(\mu) \in [0, \rho]\}$ if and only if F is absolutely monotonic on $(0, \rho)$ and such that $0 \leq F(0) \leq \lim_{\epsilon \rightarrow 0^+} F(\epsilon)$ and $\lim_{x \rightarrow \rho^-} F(x) \leq F(\rho)$.

We next examine the case where the domain of F is a symmetric closed interval $[-\rho, \rho]$. The functions preserving positivity of Hankel matrices when applied entrywise are completely characterized as follows.

Proposition 6.5. *Suppose $\rho \in (0, \infty)$ and $F : I \rightarrow \mathbb{R}$, where $I = [-\rho, \rho]$. The following are equivalent.*

- (1) $F[-]$ preserves positivity on all positive Hankel matrices with entries in I .
- (2) $F[-]$ preserves positivity on all positive Hankel matrices with entries in I , arising from moment sequences.
- (3) F is real analytic on $(-\rho, \rho)$, absolutely monotonic on $(0, \rho)$, and such that

$$F(\rho) \geq \lim_{x \rightarrow \rho^-} F(x) \quad \text{and} \quad |F(-\rho)| \leq F(\rho).$$

Proof. That (1) \implies (2) is immediate, while (2) \implies (3) follows from the extension of Theorem 2.5 given by Theorem 6.1, and the proofs of Proposition 6.3 and Lemma 4.2. Finally, if (3) holds, then (1) follows by Proposition 6.3, the Schur product theorem, and the following claim.

The only Hankel matrix in $\mathcal{P}_{N+1}([-\rho, \rho])$ with an entry $-\rho$ is the checkerboard matrix with (i, j) th entry $(-1)^{i+j}\rho$.

To prove the claim, let the rows and columns of the positive Hankel matrix A be labelled by $0, \dots, N$, and suppose $a_{ij} = -\rho$. Then $i + j$ is odd and $a_{ll} = a_{l+1, l+1} = \rho$, where $2l + 1 = i + j$. Repeatedly considering principal 2×2 minors shows that $a_{pq} = \rho$ if $p + q$ is even. Now let $m, n \in [0, N]$ be odd, with $m < n$, and denote by C the principal 3×3 minor of A corresponding to the labels $0, m$, and n . Writing

$$C = \begin{pmatrix} \rho & a_{0m} & \rho \\ a_{0m} & \rho & a_{0n} \\ \rho & a_{0n} & \rho \end{pmatrix},$$

we have that $0 \leq \det C = -\rho(a_{0m} - a_{0n})^2$, whence $a_{0m} = a_{0n}$. Taking m or n to equal $i + j$ shows that these entries equal $-\rho$, which gives the claim. \square

We end this section by considering functions preserving positivity for all matrices in $\bigcup_{N \geq 1} \mathcal{P}_N([- \rho, \rho])$. Theorem 6.1 implies that every such function F is real analytic when restricted to $(-\rho, \rho)$, and absolutely monotonic on $(0, \rho)$. The following result provides a sufficient condition for F to preserve positivity, which is also necessary if the analytic restriction is odd or even.

Proposition 6.6. *Given $\rho \in (0, \infty)$, let $I = [-\rho, \rho]$ and suppose $F : I \rightarrow \mathbb{R}$ is real analytic on $(-\rho, \rho)$, absolutely monotonic on $(0, \rho)$, and such that the limits $\lim_{x \rightarrow \rho^-} F(\pm x)$ both exist and are finite. If*

$$\left| F(-\rho) - \lim_{x \rightarrow -\rho^+} F(x) \right| \leq F(\rho) - \lim_{x \rightarrow \rho^-} F(x), \quad (6.2)$$

then $F[-]$ preserves positivity on the space of positive matrices with entries in I . The converse holds if $F|_{(-\rho, \rho)}$ is either odd or even.

The inequality (6.2) says that any jump in F at $-\rho$ is bounded above by the jump at ρ , which is non-negative.

Proof. Let g denote the continuous function on $[-\rho, \rho]$ which agrees with F on $(-\rho, \rho)$, and let the jumps $\Delta_{\pm} := F(\pm \rho) - g(\pm \rho)$. Then (6.2) is equivalent to $|\Delta_-| \leq \Delta_+$.

By the Schur product theorem and Proposition 6.3, $F[-]$ preserves positivity on $\mathcal{P}_N((-\rho, \rho])$ for all N . Now suppose $A \in \mathcal{P}_N([- \rho, \rho])$ has some entry equal to $-\rho$, where $N \geq 1$. Then the entries of A with modulus ρ form a block diagonal submatrix upon suitable relabelling of indices. This follows from the argument given in the proof of Proposition 6.3, applied to the ρ^2 -entries of $A \circ A$. Given this, and after further relabelling of indices, each block submatrix is of the form

$$\begin{pmatrix} \rho \mathbf{1}_{n_j \times n_j} & -\rho \mathbf{1}_{n_j \times m_j} \\ -\rho \mathbf{1}_{m_j \times n_j} & \rho \mathbf{1}_{m_j \times m_j} \end{pmatrix},$$

by the main result in [16], where $j = 1, \dots, r$. Then

$$F[A] = g[A] + B', \quad \text{where } B' = \bigoplus_{j=1}^k \begin{pmatrix} \Delta_+ \cdot \mathbf{1}_{n_j \times n_j} & \Delta_- \cdot \mathbf{1}_{n_j \times m_j} \\ \Delta_- \cdot \mathbf{1}_{m_j \times n_j} & \Delta_+ \cdot \mathbf{1}_{m_j \times m_j} \end{pmatrix},$$

and this is positive semidefinite, by (6.2). Thus $F[-]$ preserves $\bigcup_{N \geq 1} \mathcal{P}_N([- \rho, \rho])$.

For the converse, we show that (6.2) holds if $F[-]$ preserves positivity on just the set $\mathcal{P}_3([- \rho, \rho])$ and $F|_{(-\rho, \rho)}$ is odd or even. Note first that $\Delta_+ \geq 0$, working with 2×2 matrices as above. Next, consider the positive matrix

$$A := \begin{pmatrix} a^2/\rho & -a & a \\ -a & \rho & -\rho \\ a & -\rho & \rho \end{pmatrix},$$

and note that

$$0 \leq \lim_{a \rightarrow \rho^-} \det F[A] = \begin{vmatrix} g(\rho) & g(-\rho) & g(\rho) \\ g(-\rho) & F(\rho) & F(-\rho) \\ g(\rho) & F(-\rho) & F(\rho) \end{vmatrix} = \Delta_+(g(\rho)F(\rho) - g(-\rho)^2) - g(\rho)\Delta_-^2.$$

It follows that $\Delta_-^2 g(\rho) \leq \Delta_+^2 g(\rho)$ if $g(\rho^2) = g(-\rho)^2$, so if $g = F|_{(-\rho, \rho)}$ is odd or even. This gives the result. \square

Remark 6.7. Propositions 6.5 and 6.6 indicate the existence of functions discontinuous at $\pm\rho$ which preserve positivity for Hankel matrices, but not all matrices, in contrast to the one-sided setting of Proposition 6.3.

Indeed, if g is an odd or even function which is continuous on $[-\rho, \rho]$ and absolutely monotonic on $(0, \rho)$, define F to be equal to g on $(-\rho, \rho]$, and take $F(-\rho)$ to be any element of $(-F(\rho), F(\rho)]$. Then F preserves positivity on all Hankel matrices with entries in $[-\rho, \rho]$, but does not preserve positivity on $\bigcup_{N \geq 1} \mathcal{P}_N([-\rho, \rho])$.

7. MULTIVARIABLE GENERALIZATIONS

In this section we classify the preservers of moments arising from admissible measures in higher-dimensional Euclidean space, both in their totality and by considering their marginals.

7.1. Transformers of multivariable moment sequences. The initial generalization to higher dimensions of our characterization of moment-preserving functions raises no complications. However, the failure of Hamburger's theorem in higher dimensions, that is, the lack of a characterization of moment sequences by positivity of an associated Hankel-type kernel, means some extra work is required. Below, we isolate this additional challenge and provide a generalization of our main result.

Let μ be a non-negative measure on \mathbb{R}^d , where $d \geq 1$, which has moments of all orders; as before, such measures will be termed *admissible*. The multi-index notation

$$\mathbf{x}^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d} \quad (\mathbf{x} \in \mathbb{R}^d)$$

allows us to define the moment family

$$s_\alpha(\mu) = \int \mathbf{x}^\alpha d\mu(\mathbf{x}) \quad \forall \alpha \in \mathbb{N}_0^d,$$

where \mathbb{N}_0 denotes the set of non-negative integers. As before, we focus on measures with uniformly bounded moments, so that

$$\sup_{\alpha \in \mathbb{N}_0^d} |s_\alpha(\mu)| < \infty,$$

or, equivalently, $\text{supp}(\mu) \subset [-1, 1]^d$. In line with above, we let $\mathcal{M}(K)$ denote the set of all moment families of admissible measures supported on $K \subset \mathbb{R}^d$.

Theorem 7.1. *A function $F : \mathbb{R} \rightarrow \mathbb{R}$ maps $\mathcal{M}([-1, 1]^d)$ into itself if and only if F is absolutely monotonic and entire.*

Proof. Any admissible measure μ on $[-1, 1]$ pushes forward to an admissible measure $\tilde{\mu}$ on $[-1, 1]^d$ via the canonical embedding onto the first coordinate. If F maps $\mathcal{M}([-1, 1]^d)$ to itself then there exists an admissible measure $\tilde{\sigma}$ on $[-1, 1]^d$ such that $F(s_\alpha(\tilde{\mu})) = s_\alpha(\tilde{\sigma})$ for all $\alpha \in \mathbb{N}_0^d$, and a short calculation shows that $F[s_n(\mu)] = s_n(\sigma)$ for all $n \in \mathbb{N}_0$, where σ is the pushforward of $\tilde{\sigma}$ under the projection onto the first coordinate. Theorem 4.1 now gives that F is as claimed.

To prove the converse, we need to explore the structure of the set $\mathcal{M}([-1, 1]^d)$. Denote the generators of the semigroup \mathbb{N}_0^d by setting

$$\mathbf{1}_j := (0, \dots, 0, 1, 0, \dots, 0),$$

with 1 in the j th position. A multisequence of real numbers $(s_\alpha)_{\alpha \in \mathbb{N}_0^d}$ is the moment sequence of an admissible measure supported on $[-1, 1]^d$ if and only if the weighted Hankel-type kernels

$$(s_{\alpha+\beta}), (s_{\alpha+\beta} - s_{\alpha+\beta+2\mathbf{1}_j}), \quad 1 \leq j \leq d,$$

indexed over $\alpha, \beta \in \mathbb{N}_0^d$ are positive semidefinite [24].

Now suppose F is absolutely monotonic and entire; given a multisequence s_α subject to these positivity constraints, we have to check that the multisequence $F(s_\alpha)$ satisfies the same conditions.

As F is absolutely monotonic, Schoenberg's Theorem 2.4 gives that the kernels $(\alpha, \beta) \mapsto F(s_{\alpha+\beta})$ and $(\alpha, \beta) \mapsto F(s_{\alpha+\beta+2\mathbf{1}_j})$ are positive semidefinite. It remains to prove that the kernel

$$(\alpha, \beta) \mapsto F(s_{\alpha+\beta}) - F(s_{\alpha+\beta+2\mathbf{1}_j})$$

is positive semidefinite, for $1 \leq j \leq d$. However, as F has the Taylor expansion $F(x) = \sum_{n=0}^{\infty} c_n x^n$, with $c_n \geq 0$ for all $n \in \mathbb{N}_0$, it is sufficient to check that the kernel

$$(\alpha, \beta) \mapsto (s_{\alpha+\beta})^{on} - (s_{\alpha+\beta+2\mathbf{1}_j})^{on}$$

is positive semidefinite for any $n \in \mathbb{N}_0$. This follows from a repeated application of the Schur product theorem: if matrices A and B are such that $A \geq B \geq 0$, then

$$A^{on} \geq A^{o(n-1)} \circ B \geq A^{o(n-2)} \circ B^{o2} \geq \dots \geq B^{on}. \quad \square$$

This proof also shows that the transformers of $\mathcal{M}([-1, 1]^d)$ into $\mathcal{M}(\mathbb{R}^d)$ are the same absolutely monotonic entire functions. On the other hand, we will see in Section 8 that, in general, a mapping F as in Theorem 7.1 does not preserve the semi-algebraic supports of the underlying measures.

7.2. Transformers of moment-sequence tuples: the positive orthant case.

Our next objective is to characterize functions $F : \mathbb{R}^m \rightarrow \mathbb{R}$ which map tuples of moments $(s_k(\mu_1), \dots, s_k(\mu_m))$ arising from admissible measures on \mathbb{R} , to a moment sequence $s_k(\sigma)$ for some admissible measure σ on \mathbb{R} . This is a multivariable generalization of Schoenberg's theorem which we will demonstrate under significantly relaxed hypotheses.

More precisely, for a fixed integer $m \geq 1$, a function $F : \mathbb{R}^m \rightarrow \mathbb{R}$ acts entrywise on m -tuples of $N \times N$ matrices $(A_1 = (a_{1,ij})_{i,j=1}^N, \dots, A_m = (a_{m,ij})_{i,j=1}^N)$, so that

$$F[-] : \mathcal{P}_N(\mathbb{R})^m \rightarrow \mathbb{R}^{N \times N}, \quad F[A_1, \dots, A_m]_{ij} := F(a_{1,ij}, \dots, a_{m,ij}). \quad (7.1)$$

By the Schur product theorem, every real entire function F of the form

$$F(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}_0^m} c_\alpha \mathbf{x}^\alpha \quad \forall \mathbf{x} \in \mathbb{R}^m \quad (7.2)$$

preserves positivity on $\mathcal{P}_N(\mathbb{R})^m$ if $c_\alpha \geq 0$ for all $\alpha \in \mathbb{N}_0^m$. The reverse implication was shown by FitzGerald, Micchelli, and Pinkus in [10], and can be thought of as a multivariable version of Schoenberg's theorem. We now explain how results on several real and complex variables can be used to generalize the work in previous sections to this multivariable setting. Namely, we characterize functions mapping tuples of positive Hankel matrices into themselves. Of course, this is equivalent to mapping tuples of moment sequences of admissible measures into the same set. We also prove

an analogous result for functions defined on a bounded domain $F : (-\rho, \rho)^m \rightarrow \mathbb{R}$, in the original spirit of Schoenberg and Rudin.

First we need some notation and terminology. A function $F : \mathbb{R}^m \rightarrow \mathbb{R}$ acts on tuples of moment sequences of measures $\mathcal{M}(K_1) \times \cdots \times \mathcal{M}(K_m)$ as follows:

$$F[\mathbf{s}(\mu_1), \dots, \mathbf{s}(\mu_m)] := (F(s_k(\mu_1), \dots, s_k(\mu_m)))_{k \geq 0}, \quad (7.3)$$

where each μ_l is an admissible measure on K_l .

Given $I \subset \mathbb{R}^m$, a function $F : I \rightarrow \mathbb{R}$ is *absolutely monotonic* if F is continuous on I , and for all interior points $\mathbf{x} \in I$ and $\alpha \in \mathbb{N}_0^m$, the mixed partial derivative $D^\alpha F(\mathbf{x})$ exists and is non-negative. As usual, for a tuple $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}_0^m$, let

$$D^\alpha F(\mathbf{x}) := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}} F(x_1, \dots, x_m), \quad \text{where } |\alpha| := \alpha_1 + \cdots + \alpha_m.$$

The analogue of Bernstein's Theorem for the multivariable case is proved and put in its proper context in Bochner's book; see [7, Theorem 4.2.2].

Our first observation is the connection between functions acting on tuples of moment sequences and on the corresponding Hankel matrices. Given admissible measures μ_1, \dots, μ_m and σ supported on the real line, it is clear that

$$F[\mathbf{s}(\mu_1), \dots, \mathbf{s}(\mu_m)] = \mathbf{s}(\sigma) \iff F[H_{\mu_1}, \dots, H_{\mu_m}] = H_\sigma.$$

In particular, equality holds at each finite truncation, that is, for the corresponding leading principal $N \times N$ submatrices, for any $N \geq 1$. We will henceforth switch between moment sequences and positive Hankel matrices without further comment.

We begin by considering the case of matrices with positive entries, arising from tuples of sequences in $\mathcal{M}([0, 1])^m$. To state and prove the main result in this subsection, we require a preliminary technical result.

Lemma 7.2. *Given an integer $m \geq 1$, let \mathcal{Y}_m to be the set of all $\mathbf{y} = (y_1, \dots, y_m)^T \in (0, 1)^m$ such that the scalars*

$$\mathbf{y}^\alpha := \prod_{l=1}^m y_l^{\alpha_l}$$

are distinct for all $\alpha \in \mathbb{N}_0^m$. Then the complement of \mathcal{Y}_m in $(0, 1)^m$ has zero m -dimensional Lebesgue measure.

Proof. Let

$$\mathcal{X} := \{\mathbf{x} = \log \mathbf{y} \in (-\infty, 0)^m : \mathbf{x} \not\preceq \alpha \ \forall \alpha \in \mathbb{Z}^m \setminus \{0\}\}.$$

The complement of \mathcal{X} in $(-\infty, 0)^m$ is a countable union of hyperplanes, and so has measure zero. The result now follows since \mathcal{Y}_m is the image of \mathcal{X} under a smooth map. \square

The new notion of a *facewise absolutely monotonic function* on $[0, \infty)^m$ plays an important role in our next result. In order to define it, recall that the orthant $[0, \infty)^m$ is a convex polyhedron, and as such, is the disjoint union of the relative interiors of its faces. These faces are in bijection with subsets of $[m] := \{1, \dots, m\}$ via the mapping

$$J \mapsto \mathbb{R}_{\geq 0}^J := \{(x_1, \dots, x_m) \in [0, \infty)^m : x_l = 0 \ \forall l \notin J\}, \quad (7.4)$$

and this face has relative interior $\mathbb{R}_{> 0}^J := (0, \infty)^J \times \{0\}^{[m] \setminus J}$.

Definition 7.3. Let the integer $m \geq 1$. A function $F : [0, \infty)^m \rightarrow \mathbb{R}$ is *facewise absolutely monotonic* if, for each subset of indices $J \subset [m]$, the function F agrees on $\mathbb{R}_{>0}^J$ with an absolutely monotonic function g_J on $(0, \infty)^J$. Here and henceforth, we identify without further comment the two domains $(0, \infty)^J$ and $\mathbb{R}_{>0}^J = (0, \infty)^J \times \{0\}^{[m] \setminus J}$.

In other words, a facewise absolutely monotonic function is piecewise absolutely monotonic, with the pieces being the relative interiors of the faces of the polyhedral cone $[0, \infty)^m$. The following example illustrates this in the case $m = 2$, and will be helpful below.

Example 7.4. Let

$$F(x_1, x_2) := \begin{cases} x_1^2 + x_2^2 + 1 & \text{if } x_1, x_2 > 0, \\ 2x_1 & \text{if } x_1 > 0, x_2 = 0, \\ x_2^2 + 1 & \text{if } x_1 = 0, x_2 > 0, \\ 0 & \text{if } x_1 = x_2 = 0. \end{cases} \quad (7.5)$$

Then F is facewise absolutely monotonic, with

$$g_\emptyset = 0, \quad g_{\{1\}}(x_1) = 2x_1, \quad g_{\{2\}}(x_2) = x_2^2 + 1, \quad \text{and} \quad g_{\{1,2\}}(x_1, x_2) = x_1^2 + x_2^2 + 1.$$

Notice that in this example, and, in fact, for every facewise absolutely monotonic function, the function g_J extends to an absolutely monotonic function on the closure $[0, \infty)^J$ of its domain, for all $J \subset [m]$. We denote this extension by \tilde{g}_J .

Furthermore, note that in Equation (7.5), the functions \tilde{g}_J satisfy a form of monotonicity that is compatible with the partial order on their labels:

$$K \subset J \subset [m] \implies 0 \leq \tilde{g}_K \leq \tilde{g}_J \quad \text{on } \mathbb{R}_{\geq 0}^K \simeq [0, \infty)^K. \quad (7.6)$$

A word of caution: while $\tilde{g}_{\{1\}}(x_1) \leq \tilde{g}_{\{1,2\}}(x_1, 0)$ for all $x_1 \geq 0$, it is not true that the difference of these functions is absolutely monotonic on $[0, \infty)$.

With this definition and example in hand, together with Lemma 7.2, we can now characterize the preservers of moment sequences in $\mathcal{M}([0, 1]^m)$.

Theorem 7.5. Suppose $m \geq 1$ and $F : [0, \infty)^m \rightarrow \mathbb{R}$, and fix $\mathbf{y} = (y_1, \dots, y_m)^T \in \mathcal{Y}_m$, as in Lemma 7.2. The following are equivalent.

- (1) $F[-]$ maps $\mathcal{M}(\{1, y_1\}) \times \dots \times \mathcal{M}(\{1, y_m\})$ into $\mathcal{M}(\mathbb{R})$, and

$$F((a_1, \dots, a_m))F((b_1, \dots, b_m)) \geq F\left((\sqrt{a_1 b_1}, \dots, \sqrt{a_m b_m})\right)^2$$

for all $a_1, \dots, a_m, b_1, \dots, b_m \geq 0$.

- (2) $F[-]$ maps $\mathcal{M}([0, 1]^m)$ into $\mathcal{M}([0, 1])$.

- (3) F is facewise absolutely monotonic, and the functions $\{g_J : J \subset [m]\}$ satisfy the monotonicity condition (7.6).

Reformulating this result, as in the one-dimensional case above, it suffices to work only with Hankel matrices of rank at most two. Moreover, Theorem 3.2 is precisely Theorem 7.5 when $m = 1$. The proof builds on Theorem 3.2; however, the higher dimensionality introduces several additional challenges.

A large part of Theorem 7.5 can be deduced from the following reformulation on the open cell in the positive orthant.

Theorem 7.6. Fix $\rho \in (0, \infty]$, an integer $m \geq 1$ and a point $\mathbf{y} = (y_1, \dots, y_m)^T \in \mathcal{Y}_m$, as in Lemma 7.2. For $1 \leq l \leq m$ and $N \geq 1$, let

$$\mathbf{u}_{l,N} := (1, y_l, \dots, y_l^{N-1})^T,$$

$$\text{and } \mathbf{H}_l^+(N) := \{a\mathbf{1}_{N \times N} + b\mathbf{u}_{l,N}\mathbf{u}_{l,N}^T : a \in (0, \rho), b \in [0, \rho - a]\}.$$

If the function $F : (0, \rho)^m \rightarrow \mathbb{R}$ is such that $F[-]$ preserves positivity on $\mathcal{P}_2^1((0, \rho))^m$ and on $\mathbf{H}_1^+(N) \times \dots \times \mathbf{H}_m^+(N)$, for all $N \geq 1$, then F is absolutely monotonic and is the restriction of an analytic function on $D(0, \rho)^m$.

Note that the matrices in $\mathbf{H}_l^+(N)$ are precisely the truncated moment matrices of admissible measures supported on $\{1, y_l\}$.

Proof. We begin by recording a few basic properties of F . First, either F is identically zero, or it is everywhere positive on $(0, \rho)^m$; this may be shown similarly to the proof of Theorem 3.5. Also, given $\mathbf{c} = (c_1, \dots, c_m)^T \in (0, \rho)^m$, the function g such that

$$g(\mathbf{x}) := F(\mathbf{x} + \mathbf{c}) \quad \forall \mathbf{x} \in (0, \rho - c_1) \times \dots \times (0, \rho - c_m)$$

satisfies the same hypotheses as F , but with ρ replaced by $\rho - c_l$ in each $\mathbf{H}_l^+(N)$, and with $\mathcal{P}_2^1((0, \rho))^m$ replaced by $\mathcal{P}_2^1((0, \rho - c_1)) \times \dots \times \mathcal{P}_2^1((0, \rho - c_m))$. Furthermore, using only tuples in $\mathcal{P}_2^1((0, \rho))$, as well as the hypotheses, one can argue as in the proof of Theorem 3.5, and show that F is continuous on $(0, \rho)^m$.

Next, as in the proof of [10, Theorem 2.1], a mollifier argument reduces the problem to considering only smooth F . We now follow the proof of [10, Proposition 2.5], but with suitable modifications imposed by the weaker hypotheses.

Given $r \geq 0$, we take $N \geq \binom{r+m}{m}$, and let $\mathbf{y}'_l := (1, y_l, \dots, y_l^{N-1})^T$ for $1 \leq l \leq m$. Fix some $\mathbf{c} \in (0, \rho)^m$, choose $b_l \in (0, \rho - c_l)$ for all l and let

$$A_l := c_l \mathbf{1}_{N \times N} + b_l \mathbf{y}'_l \mathbf{y}'_l{}^T \in \mathbf{H}_l^+(N),$$

so that $F[A_1, \dots, A_m] \in \mathcal{P}_N(\mathbb{R})$. We now use Lemma 7.2: since $\mathbf{y} \in \mathcal{Y}_m$ and $N \geq \binom{r+m}{m}$ by assumption, for each $\beta \in \mathbb{N}_0^m$ with $|\beta| \leq r$ we can choose $\mathbf{v}_\beta \in \mathbb{R}^N$ such that

$$\mathbf{v}_\beta \perp (1, \mathbf{y}^\alpha, \mathbf{y}^{2\alpha}, \dots, \mathbf{y}^{(N-1)\alpha})^T \quad \forall \alpha \in \mathbb{N}_0^m \setminus \{\beta\} \text{ with } |\alpha| \leq r,$$

and $(1, \mathbf{y}^\beta, \dots, \mathbf{y}^{(N-1)\beta})\mathbf{v}_\beta = 1$. An application of Taylor's theorem (similar to its use in Proposition 3.12 or [10, Proposition 2.5]) now gives that the derivative $D^\beta F(\mathbf{c}) \geq 0$. Thus F is absolutely monotonic on $(0, \rho)^m$, and Schoenberg's observation [27, Theorem 5.2] implies that F is the restriction to $(0, \rho)^m$ of an analytic function on $D(0, \rho)^m$. \square

With this result in hand, we can now prove the theorem.

Proof of Theorem 7.5. Clearly, (2) \implies (1).

We will show (1) \implies (3) by induction on m . As noted above, the case $m = 1$ is precisely Theorem 3.2. For the induction step, we first restrict F to the relative interior of any face of the polyhedron $[0, \infty)^m$, say $\mathbb{R}_{>0}^J$ for some $J \subset [m]$. The induction hypothesis and Theorem 7.6 give that F is facewise absolutely monotonic, so $F \equiv g_J$ on $(0, \infty)^J$, with g_J absolutely monotonic. To see that (7.6) holds, we claim that, for all subsets $L \subset K \subsetneq J \subset [m]$,

$$\tilde{g}_K(\mathbf{x}) \leq \tilde{g}_J(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}_{>0}^L \subset [0, \infty)^m.$$

For ease of exposition, we show this for an illustrative example; the general case follows with minimal modification. Suppose $J = \{1, 2, 3\}$, $K = \{1, 2\}$, and $L = \{1\}$. For any $(x_1, 0, 0) \in \mathbb{R}_{>0}^L$, we set

$$(a_1, a_2, a_3) := (x_1, x_2, x_3) \quad \text{and} \quad (b_1, b_2, b_3) := (x_1, x_2, 0),$$

where $x_2 > 0$ and $x_3 > 0$. By hypothesis (1), it follows that

$$\tilde{g}_J(x_1, x_2, x_3) \tilde{g}_K(x_1, x_2, 0) \geq \tilde{g}_K(x_1, x_2, 0)^2,$$

and taking limits as $x_2 = x_{K \setminus L} \rightarrow 0^+$ and $x_3 = x_{J \setminus K} \rightarrow 0^+$, we have that

$$\tilde{g}_J(x_1, 0, 0) \tilde{g}_K(x_1, 0, 0) \geq \tilde{g}_K(x_1, 0, 0)^2,$$

and so (7.6) holds as required.

Finally, to show that (3) \implies (2), given positive Hankel matrices A_1, \dots, A_m arising from moment sequences in $\mathcal{M}([0, 1])$, let

$$J := \{l \in [m] : a_{l,11} > 0\} \quad \text{and} \quad K := \{l \in [m] : a_{l,22} > 0\}.$$

Note that $K \subset J \subset [m]$. Recalling that the only Hankel matrices arising from $\mathcal{M}([0, 1])$ and having zero entries are of the form $H_{a\delta_0}$ for some $a \geq 0$, we may write

$$F[A_1, \dots, A_m] = (g_J(a_{l,11} : l \in J) - g_K(a_{l,11} : l \in K)) H_{\delta_0} + g_K[A_l : l \in K]. \quad (7.7)$$

For example, given $a, b, c, d > 0$, we have that

$$\begin{aligned} F \left[\begin{pmatrix} a & b \\ b & c \end{pmatrix}, \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right] &= \begin{pmatrix} g_{\{1,2\}}(a, d) & g_{\{1\}}(b) \\ g_{\{1\}}(b) & g_{\{1\}}(c) \end{pmatrix} \\ &= (g_{\{1,2\}}(a, d) - g_{\{1\}}(a)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + g_{\{1\}} \left[\begin{pmatrix} a & b \\ b & c \end{pmatrix} \right]. \end{aligned}$$

The proof concludes by observing that both terms in the right-hand side of (7.7) are positive semidefinite, by the Schur product theorem and hypothesis (3):

$$\begin{aligned} g_J(a_{l,11} : l \in J) &\geq \lim_{a_{l,11} \rightarrow 0^+ \forall l \in J \setminus K} g_J(a_{l,11} : l \in J) = \tilde{g}_J(a_{l,11} : l \in K) \\ &\geq g_K(a_{l,11} : l \in K). \end{aligned} \quad \square$$

As Theorem 7.5 shows, the notion of facewise absolutely monotone maps on $[0, \infty)^m$ is a refinement of absolute monotonicity, emerging from the study of positivity preservers of tuples of moment sequences, i.e., the Hankel matrices arising from them. If, instead, one studies maps preserving positivity on tuples of all positive semidefinite matrices, or even all Hankel matrices, then this richer class of maps does not arise.

Proposition 7.7. *Suppose $\rho \in (0, \infty]$ and $F : I^m \rightarrow \mathbb{R}$, where $I = [0, \rho]$. The following are equivalent.*

- (1) $F[-]$ preserves positivity on the space of m -tuples of positive Hankel matrices with entries in I .
- (2) F is absolutely monotonic on I^m .
- (3) $F[-]$ preserves positivity on the space of m -tuples of all positive matrices with entries in I .

Proof. Clearly (2) \implies (3) \implies (1). Now suppose (1) holds. By Theorem 7.6, F is absolutely monotonic on the domain $(0, \rho)^m$, and agrees there with an analytic function $g : D(0, \rho)^m \rightarrow \mathbb{C}$. We now claim $F \equiv g$ on I^m . The proof is by induction on m , with the $m = 1$ case shown in Proposition 6.3.

Suppose $m > 1$, and let $\mathbf{c} = (c_1, \dots, c_m) \in I^m \setminus (0, \rho)^m$. Note that at least one coordinate of \mathbf{c} is zero. We choose $\mathbf{u}_n = (u_{1,n}, \dots, u_{m,n}) \in (0, \rho)^m$ such that $\mathbf{u}_n \rightarrow \mathbf{c}$, and we wish to show that $F(\mathbf{u}_n) = g(\mathbf{u}_n) \rightarrow F(\mathbf{c})$. Let

$$H := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad A_{l,n} := \begin{cases} u_{l,n} \mathbf{1}_{3 \times 3} & \text{if } c_l > 0, \\ u_{l,n} H & \text{if } c_l = 0. \end{cases}$$

Using (1) and the induction hypothesis for the (1, 2) and (2, 1) entries, it follows that

$$\lim_{n \rightarrow \infty} F[A_{1,n}, \dots, A_{m,n}] = \begin{pmatrix} g(\mathbf{c}) & F(\mathbf{c}) & g(\mathbf{c}) \\ F(\mathbf{c}) & g(\mathbf{c}) & g(\mathbf{c}) \\ g(\mathbf{c}) & g(\mathbf{c}) & g(\mathbf{c}) \end{pmatrix} \in \mathcal{P}_3.$$

Computing the determinants of the leading principal minors gives

$$g(\mathbf{c}) \geq 0, \quad g(\mathbf{c}) \geq |F(\mathbf{c})|, \quad \text{and} \quad -g(\mathbf{c})(g(\mathbf{c}) - F(\mathbf{c}))^2 \geq 0.$$

Hence $F(\mathbf{c}) = g(\mathbf{c})$, and the proof is complete. \square

7.3. Transformers of moment-sequence tuples: the general case. Having resolved the characterization problem for functions defined on the positive orthant, we now work over the whole of \mathbb{R}^m .

Theorem 7.8. *Suppose $F : \mathbb{R}^m \rightarrow \mathbb{R}$ for some integer $m \geq 1$. The following are equivalent.*

- (1) *Applied entrywise, F maps $\mathcal{M}([-1, 1])^m$ into $\mathcal{M}(\mathbb{R})$.*
- (2) *Applied entrywise, F maps $\mathcal{P}_N(\mathbb{C})^m$ into $\mathcal{P}_N(\mathbb{C})$ for all $N \geq 1$.*
- (3) *The function F is entire, and absolutely monotonic on $[0, \infty)^m$.*

In particular, akin to the one-variable case, Theorem 7.8 strengthens the multivariable analogue of Schoenberg's theorem in [10] by using only Hankel matrices arising from tuples of moment sequences. Moreover, akin to the $m = 1$ case, the proof reveals that one only requires Hankel matrices of rank at most 3.

Corollary 7.9. *The hypotheses in Theorem 7.8 are also equivalent to the following.*

- (4) *Fix $\epsilon > 0$ and $u_0 \in (0, 1)$. For $v \in (0, 1 + \epsilon)$, let $\mathcal{M}_v := \mathcal{M}(\{-1, v, 1\})$, and let*

$$\mathcal{M}_{[u_0]} := \bigcup_{s_1 \in \{-1, 0, 1\}, s_2 \in \{-u_0, 0, u_0\}} \mathcal{M}(\{s_1, s_2\}).$$

$$\text{Then } F[-] \text{ maps } \mathcal{M}_{[u_0]}^m \cup \bigcup_{v_1, \dots, v_m \in (0, 1 + \epsilon)} \mathcal{M}_{v_1} \times \dots \times \mathcal{M}_{v_m} \text{ into } \mathcal{M}(\mathbb{R}).$$

As the reader will observe, hypothesis (4) is stronger, even in the one-dimensional case, than the corresponding hypothesis in Theorem 4.1. As the proof shows, these extra assumptions are required to obtain continuity on every orthant and on 'walls' between orthants, as well as real analyticity on one-parameter curves.

Proof of Theorem 7.8 and Corollary 7.9. Clearly (3) \implies (2) \implies (1) \implies (4) by the Schur product theorem. Thus, we will assume (4) and obtain (3). By Theorem 7.6, the function F is absolutely monotonic on the open positive orthant $(0, \infty)^m$, and equals the restriction to $(0, \infty)^m$ of an entire function $g : \mathbb{C}^m \rightarrow \mathbb{C}$. We now show that $F \equiv g$ on all of \mathbb{R}^m . The proof follows the $m = 1$ case in Section 4; for ease of exposition, we break it up into steps.

Step 1. We first prove F is locally bounded. This follows by using $\mathcal{M}_2(\{-1\})^m$, as in the proof of Lemma 4.2. As above, this gives that

$$F[-] : \mathcal{M}_{[u_0]}^m \cup \mathcal{M}_{v_1} \times \cdots \times \mathcal{M}_{v_m} \rightarrow \mathcal{M}([-1, 1]), \quad \forall v_1, \dots, v_m \in (-1, 1). \quad (7.8)$$

Step 2. Next, we show that F is continuous on \mathbb{R}^m . The first objective is to show continuity of F inside each open orthant of \mathbb{R}^m . Given nonzero c_1, \dots, c_m , and any sequence $\{(v_{1,n}, \dots, v_{m,n}) : n \geq 1\} \subset \mathbb{R}^m$ converging to the origin, let

$$a_{l,n} := |c_l| + \frac{\operatorname{sgn}(c_l)u_0}{u_0 - u_0^3}v_{l,n} \quad \text{and} \quad \mu_{l,n} := a_{l,n}\delta_{\operatorname{sgn}(c_l)} + \frac{|v_{l,n}|}{u_0 - u_0^3}\delta_{-\operatorname{sgn}(v_{l,n})u_0} \quad (7.9)$$

for $1 \leq l \leq m$. Note that, for sufficiently large n , the sequence $\mathbf{s}(\mu_{l,n}) \in \mathcal{M}_{[u_0]}$.

We now follow the proof of Proposition 4.4. Suppose that $F[H_{\mu_{1,n}}, \dots, H_{\mu_{m,n}}] = H_{\sigma_n}$ for some admissible measure $\sigma_n \in \operatorname{Meas}^+([-1, 1])$, for every $n \geq 1$. The polynomials $p_{\pm}(t) := (1 \pm t)(1 - t^2)$ are non-negative on $[-1, 1]$, so, by (3.1),

$$\begin{aligned} \int_{-1}^1 p_{\pm}(t) d\sigma_n &\geq 0 \\ \implies F(s_0(\mu_{l,n})_{l=1}^m) - F(s_2(\mu_{l,n})_{l=1}^m) &\geq |F(s_1(\mu_{l,n})_{l=1}^m) - F(s_3(\mu_{l,n})_{l=1}^m)|. \end{aligned} \quad (7.10)$$

Computing the moments of $\mu_{l,n}$ gives the following:

$$\begin{aligned} s_0(\mu_{l,n}) &= |c_l| + \frac{\operatorname{sgn}(c_l)u_0 + \operatorname{sgn}(v_{l,n})}{u_0 - u_0^3}v_{l,n}, & s_1(\mu_{l,n}) &= c_l, \\ s_2(\mu_{l,n}) &= |c_l| + \frac{\operatorname{sgn}(c_l)u_0 + \operatorname{sgn}(v_{l,n})u_0^2}{u_0 - u_0^3}v_{l,n}, & s_3(\mu_{l,n}) &= c_l + v_{l,n}. \end{aligned} \quad (7.11)$$

As $n \rightarrow \infty$, by the continuity of F in $(0, \infty)^m$, the left-hand side of (7.10) goes to zero, whence so does the right-hand side, which is $|F(c_1, \dots, c_m) - F(c_1 + v_{1,n}, \dots, c_m + v_{m,n})|$. This proves the continuity of F at (c_1, \dots, c_m) , so in every open orthant of \mathbb{R}^m .

To conclude this step, we show F is continuous on the boundary of the orthants, that is, on the union of the coordinate hyperplanes:

$$\mathcal{Z} := \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_1 \cdots x_m = 0\}.$$

The proof is by induction on m , with the case $m = 1$ shown in Proposition 4.4. For general $m \geq 2$, by the induction hypothesis F is continuous when restricted to \mathcal{Z} . It remains to prove F is continuous at a point $\mathbf{c} = (c_1, \dots, c_m) \in \mathcal{Z}$ when approached along a sequence $\{(c_1 + v_{1,n}, \dots, c_m + v_{m,n}) : n \geq 1\}$ which lies in the interior of some orthant in \mathbb{R}^m . Repeating the computations for (7.11), with the same sequences $a_{l,n}$ and $\mu_{l,n}$, and the polynomials $p_{\pm}(t)$, we note that if $c_l \neq 0$ then $s_0(\mu_{l,n}) > 0$ and $s_2(\mu_{l,n}) > 0$ for all sufficiently large n , while if $c_l = 0$ then $s_0(\mu_{l,n}) > 0$ and $s_2(\mu_{l,n}) > 0$ for all n , since $c_l + v_{l,n} \neq 0$ by assumption. Therefore, in all cases, the left-hand side of (7.10) eventually equals $F(\mathbf{u}_n) - F(\mathbf{u}'_n)$, with \mathbf{u}_n and \mathbf{u}'_n in the positive open

orthant $(0, \infty)^m$, and both converging to $|\mathbf{c}| := (|c_1|, \dots, |c_m|)$. Since $F \equiv g$ on $(0, \infty)^m$ for some entire function g , so (7.10) gives that

$$\lim_{n \rightarrow \infty} |F(\mathbf{c}) - F(c_1 + v_{1,n}, \dots, c_m + v_{m,n})| \leq \lim_{n \rightarrow \infty} F(\mathbf{u}_n) - F(\mathbf{u}'_n) = g(|\mathbf{c}|) - g(|\mathbf{c}|) = 0.$$

It follows that F is continuous at all $\mathbf{c} \in \mathcal{Z}$, and hence on all of \mathbb{R}^m , as claimed.

Step 3. The next step in the proof is to show that it suffices to consider F to be smooth. This is achieved using a mollifier argument, exactly as in the one-variable situation. As observed in Remark 4.5, to use this argument we require every domain in Corollary 7.9 to contain 1.

Step 4. We now claim F is real analytic in every one-parameter space $\mathbf{c} + e^{-\mathbb{R}\mathbf{v}} \subset \mathbb{R}^m$ at the point $\mathbf{c} + \mathbf{1}$, where $\mathbf{1} = (1, \dots, 1)$ and $e^{-x\mathbf{v}} := (e^{-xv_1}, \dots, e^{-xv_m})$ for every $x \in \mathbb{R}$. Indeed, fix $s > 0$, and let $|\mathbf{c}| := (|c_1|, \dots, |c_m|)$,

$$\mu_{l,s} := |c_l| \delta_{\text{sgn}(c_l)} + e^{-xv_l} \delta_{e^{-sv_l}} \quad \text{and} \quad p_{\pm,n}(t) := (1 \pm t)(1 - t^2)^n,$$

where $n \geq 0$ and $1 \leq l \leq m$. We note that working with moment sequences of $\mu_{l,s}$ may require working with \mathcal{M}_v for $v > 1$, if $v_l < 0$ for some l .

Returning to the proof, as $p_{\pm,n}(t) \geq 0$ for all $t \in [-1, 1]$ and all $n \geq 0$, applying (3.1) gives that

$$\left| \sum_{k=0}^n \binom{n}{k} (-1)^k F(|\mathbf{c}| + e^{-(x+2ks)\mathbf{v}}) \right| \geq \left| \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} F(\mathbf{c} + e^{-(x+(2k+1)s)\mathbf{v}}) \right|.$$

Letting $H_{\mathbf{c},\mathbf{v}}(x) := F(\mathbf{c} + e^{-x\mathbf{v}})$, dividing both sides of this inequality by s^n and then taking $s \rightarrow 0^+$, it follows that

$$\left| \frac{d^n}{dx^n} H_{|\mathbf{c}|,\mathbf{v}}^{(n)}(x) \right| \geq \left| \frac{d^n}{dx^n} H_{\mathbf{c},\mathbf{v}}^{(n)}(x) \right|.$$

These estimates prove that the function $x \mapsto F(\mathbf{c} + \mathbf{1} + [e^{-x\mathbf{v}} - \mathbf{1}])$ is real analytic in a neighborhood of $x = 0$, as claimed.

We now complete the proof. The real-analytic local diffeomorphism

$$T : (u_1, \dots, u_m) \mapsto (e^{u_1} - 1, e^{u_2} - 1, \dots, e^{u_m} - 1)$$

maps $0 \in \mathbb{R}^m$ to $0 \in \mathbb{R}^m$, and the function

$$\mathbf{u} \mapsto F(\mathbf{c} + \mathbf{1} + T(\mathbf{u}))$$

is smooth and real analytic along every straight line passing through the origin. Standard criteria for real analyticity (see e.g. [2, Theorem 5.5.33]) now give that F is real analytic at the point $\mathbf{c} + \mathbf{1}$, hence at every point of \mathbb{R}^m .

Finally, we recall that F agrees on $(0, \infty)^m$ with the entire function $g : \mathbb{C}^m \rightarrow \mathbb{C}$. Since $F : \mathbb{R}^m \rightarrow \mathbb{R}$ is real analytic, so $F = g|_{\mathbb{R}^m}$ and the proof is complete. \square

Remark 7.10. As Step 2 in the proof shows, we may replace $\mathcal{M}_{[u_0]}^m$ in hypothesis (4) of Corollary 7.9 by $\mathcal{M}_{[u_1]} \times \dots \mathcal{M}_{[u_m]}$ for any $u_1, \dots, u_m \in (0, 1)$.

Remark 7.11. Akin to the one-dimensional case, one may now show that Theorems 7.5 and 7.8 hold more generally for tuples of measures with bounded mass. More precisely, one should fix $\rho_1, \dots, \rho_m \in (0, \infty)$ and work with tuples of admissible measures (μ_1, \dots, μ_m) supported in $[-1, 1]$ and such that $s_0(\mu_l) < \rho_l$ for $1 \leq l \leq m$, whence $s_k(\mu_l) < \rho_l$ for every $k \geq 0$ and all such l . As discussed in the Introduction, this

explains how our results unify and strengthen the Schoenberg–Rudin theorem and the FitzGerald–Micchelli–Pinkus result for positivity preservers.

To prove Theorem 7.5 for $F : I_1 \times \cdots \times I_m \rightarrow \mathbb{R}$, where $I_l = [0, \rho_l)$, one should first define facewise absolutely monotonic maps on $I_1 \times \cdots \times I_m$ using the relative interiors of the faces cut out by the same functionals as for $[0, \infty)^m$. The existing proof for the case $\rho_1 = \cdots = \rho_m = \infty$ goes through with minimal modifications, including to Theorem 7.6. The same is true for proving Theorem 7.8 with the domain $(-\rho_1, \rho_1) \times \cdots \times (-\rho_m, \rho_m)$ in place of \mathbb{R}^m .

8. LAPLACE-TRANSFORM INTERPRETATIONS

When speaking about completely monotonic or absolutely monotonic functions one cannot leave aside Laplace transforms. We briefly touch the subject below, in connection with our theme.

Let F be an absolutely monotonic function on $(0, \infty)$, and let μ and σ be admissible measures supported on $[0, 1]$ such that

$$F(s_k(\mu)) = s_k(\sigma) \quad \forall k \geq 0. \quad (8.1)$$

By the change of variable $x = e^{-t}$, we can push forward the restriction of the measure μ to $(0, 1]$ to a measure μ_1 on $[0, \infty)$, and similarly for σ . Thus, with the possible loss of zeroth-order moments, we obtain

$$s_k(\mu) = \int_0^\infty e^{-kt} d\mu_1(t) \quad \text{and} \quad s_k(\sigma) = \int_0^\infty e^{-kt} d\sigma_1(t).$$

If \mathcal{L} denotes the Laplace transform, so that

$$\mathcal{L}\nu(z) = \int_0^\infty e^{-tz} d\nu(t),$$

then $\mathcal{L}\nu$ is a complex-analytic function in the open half-plane $\mathbb{C}^+ := \{z \in \mathbb{C} : \Re z > 0\}$. Our assumption (8.1) becomes

$$F(\mathcal{L}\mu_1(k)) = \mathcal{L}\sigma_1(k) \quad \forall k \geq 1,$$

and a classical observation due to Carlson [8] implies that

$$F(\mathcal{L}\mu_1(z)) = \mathcal{L}\sigma_1(z) \quad \forall z \in \mathbb{C}^+.$$

More precisely, Carlson’s Theorem asserts that a bounded analytic function in the right half-plane is identically zero if it vanishes at all positive integers. The proof relies on the Phragmén–Lindelöf principle [23]; see also [5] or [31, §5.8] for more details.

In this section, we will show some results from the interplay between the Laplace transform and functions which transform positive Hankel matrices.

For point masses, the situation is rather straightforward. If $\mu = \delta_{e^{-a}}$ for some point $a \in [0, \infty)$, and $F(x) = \sum_{n=0}^\infty c_n x^n$, then

$$F(\mathcal{L}\mu(z)) = F(e^{-az}) = \sum_{n=0}^\infty c_n e^{-anz} = \mathcal{L}\sigma_1(z),$$

where

$$\sigma_1 = \sum_{n=0}^\infty c_n \delta_{an} \quad \text{and} \quad \sigma = \sum_{n=0}^\infty c_n \delta_{e^{-an}}.$$

More generally, if μ has countable support, then the transform $F[-]$ will yield a measure with countable support also. A strong converse to this is the following result.

Proposition 8.1. *Let $a \in (0, 1)$ and suppose the function $F : x \mapsto \sum_{n=0}^{\infty} c_n x^n$ is absolutely monotonic on $(0, \infty)$. The following are equivalent.*

(1) *There exists an admissible measure μ on $[0, 1]$ such that*

$$F(s_k(\mu)) = a^k \quad \forall k \geq 0.$$

(2) *$F(x) = x^N$ for some $N \geq 1$, and $\mu = \delta_{a^{1/N}}$.*

Proof. That (2) \implies (1) is clear. Now suppose (1) holds. Setting $\psi(t) := -\log t$,

$$s_k(\mu) = \int_0^1 x^k d\mu(t) = \int_0^\infty e^{-kt} d\nu(t) = \mathcal{L}\nu(k) \quad \forall k \geq 0,$$

where $\nu := \psi_*\mu$ is the push-forward of μ under ψ . If $a = e^{-\lambda}$ for some $\lambda > 0$, then, by assumption,

$$F(\mathcal{L}\nu(k)) = e^{-\lambda k} \quad \forall k \geq 1.$$

and, by Carlson's Theorem,

$$F(\mathcal{L}\nu(z)) = e^{-\lambda z} \quad \forall z \in \mathbb{C}^+. \quad (8.2)$$

In view of Bernstein's theorem [32, Chapter IV, Theorem 12a], the function $\mathcal{L}\nu$ is completely monotonic on $[0, \infty)$. Now, since the composition of an absolutely monotonic function and a completely monotonic function is completely monotonic, so

$$z \mapsto (\mathcal{L}\nu(z))^k = \left(\int_0^\infty e^{-zt} d\nu(t) \right)^k$$

is completely monotonic on $[0, \infty)$ for all $k \in \mathbb{N}_0$. Thus, by another application of Bernstein's theorem, there exists an admissible measure ν_k on $[0, \infty)$ such that

$$(\mathcal{L}\nu(z))^k = \int_0^\infty e^{-zt} d\nu_k(t) \quad \forall z \in \mathbb{C}^+.$$

Using the above expression, we can rewrite (8.2) as

$$F(\mathcal{L}\nu(z)) = \sum_{n=0}^{\infty} c_n (\mathcal{L}\nu_n)(z) = \mathcal{L} \left(\sum_{n=0}^{\infty} c_n \nu_n \right) (z) = e^{-\lambda z} = (\mathcal{L}\delta_\lambda)(z),$$

and, by the uniqueness principle for Laplace transforms, we conclude that

$$\sum_{n=0}^{\infty} c_n \nu_n = \delta_\lambda.$$

Now, let A be any measurable subset of $[0, \infty)$ that does not contain λ . Then,

$$\left(\sum_{n=0}^{\infty} c_n \nu_n \right) (A) = \delta_\lambda(A) = 0.$$

Since $c_n \geq 0$, it follows that $c_n \nu_n(A) = 0$ for all measurable sets A not containing λ , and all $n \in \mathbb{N}_0$. Hence, either $c_n = 0$, or $\nu_n = \delta_\lambda$. Moreover, $\sum_{n=0}^{\infty} c_n = 1$.

Now, suppose $c_n \neq 0$ for some n . By the above argument, we must have $\nu_n = \delta_\lambda$. Thus,

$$\mathcal{L}\nu_n(z) = \left(\int_0^\infty e^{-zt} d\nu(t) \right)^n = e^{-\lambda z} \quad \forall z \in \mathbb{C}^+.$$

Equivalently,

$$\int_0^\infty e^{-zt} d\nu(t) = e^{-\lambda z/n},$$

and applying the uniqueness principle for the Laplace transform one more time gives that $\nu = \delta_{\lambda/n}$. Hence $c_n \neq 0$ for at most one n , say for $n = N$, so $F(x) = x^N$ and $\nu = \delta_{\lambda/N}$. Finally, since $\nu = \psi_*\mu$, we conclude that $\mu = \delta_{a^{1/N}}$, as claimed. \square

9. CONCLUDING REMARKS

Tables 9.1 and 9.2 below summarize the results proved in this article. In the one-variable setting, we identified the positivity preservers acting on (i) all matrices, and (ii) all Hankel matrices, in the course of classifying such functions acting on (iii) moment sequences, i.e., all Hankel matrices arising from moment sequences of measures supported on $[-1, 1]$. Characterizations for all three classes of matrices were obtained with the additional constraint that the entries of the matrices lie in $(0, \rho)$, $(-\rho, \rho)$, and $[0, \rho]$, where $\rho \in (0, \infty]$.

	$F[-]$ preserves positivity on:		
Domain I , $\rho \in (0, \infty]$	$\cup_{N \geq 1} \mathcal{P}_N(I)$	$\mathcal{H}^+(I)$	$\mu \in \mathcal{M}([0, 1])$ or $\mathcal{M}([-1, 1])$, $s_0(\mu) \in I \cap [0, \infty)$
$(0, \rho)$	Theorems 3.4, 3.8	Theorems 3.5, 3.8	Theorems 3.5, 3.8
$[0, \rho)$	Proposition 6.3	Proposition 6.3	Theorems 3.2, 6.1
$(-\rho, \rho)$	Theorem 2.4	Theorems 4.1, 6.1	Theorems 4.1, 6.1

TABLE 9.1. The one-variable case.

We then extended each of the results in Table 9.1 to apply to functions acting on tuples of positive matrices or moment sequences: see Table 9.2.

	$F[-]$ preserves positivity on m -tuples of elements in:		
Domain I , $\rho \in (0, \infty]$	$\cup_{N \geq 1} \mathcal{P}_N(I)$	$\mathcal{H}^+(I)$	$\mu \in \mathcal{M}([0, 1])$ or $\mathcal{M}([-1, 1])$, $s_0(\mu) \in I \cap [0, \infty)$
$(0, \rho)$	Theorem 7.6	Theorem 7.6	Theorem 7.6
$[0, \rho)$	Proposition 7.7	Proposition 7.7	Theorem 7.5
$(-\rho, \rho)$	Theorem 7.8 (see [10] for $\rho = \infty$)	Theorem 7.8	Theorem 7.8

TABLE 9.2. The multivariable case.

We point out that, in the one-variable setting, we do more than is recorded in Table 9.1, since our results cover the closed-interval settings of $[0, \rho]$ and $[-\rho, \rho]$ for $\rho < \infty$; see Section 6. In this direction, note also that the multivariable case may contain products of open and closed intervals, but it would be rather cumbersome, and somewhat artificial, to consider them all. We do not pursue this in the present work.

In all of the above contexts, with the exception of functions on $[0, \rho]^m$ that preserve moment sequences (i.e., the (2, 3) entry in both tables), the characterizations are uniform: all such positivity preservers are necessarily analytic on the domain and absolutely monotonic on the closed positive orthant. The converse result holds trivially by the Schur product theorem. The one exceptional case reveals a richer family of “facewise absolutely monotonic maps”; see Section 7.2.

We have also improved on all of the above results, by significantly weakening the hypotheses required to obtain absolute monotonicity.

List of symbols. For the convenience of the reader, we list some of the symbols used in this paper.

- Given a subset $I \subset \mathbb{R}$, $\mathcal{P}_N^k(I)$ is the set of positive semidefinite $N \times N$ matrices with entries in I and of rank at most k . We let $\mathcal{P}_N(I) := \mathcal{P}_N^N(I)$.
- $\mathcal{H}^+(I)$ denotes the set of positive semidefinite Hankel matrices of arbitrary dimension with entries in I .
- $H^{(1)}$ denotes the truncation of a possibly semi-infinite matrix H obtained by excising the first column.
- $F[H]$ is the result of applying F to each entry of the matrix H .
- For $K \subset \mathbb{R}$, we denote by $\text{Meas}^+(K)$ the set of admissible measures, i.e., non-negative measures μ supported on K and admitting moments of all orders.
- The k th moment of a measure μ is denoted by $s_k(\mu)$; the corresponding moment sequence is $\mathbf{s}(\mu) := (s_k(\mu))_{k \geq 0}$. The associated Hankel moment matrix H_μ has (i, j) entry $s_{i+j}(\mu)$. In particular, the moment sequence of μ is the leading row and column of H_μ .
- Given $K \subset \mathbb{R}$, $\mathcal{M}(K)$ denotes the set of moment sequences associated to elements of $\text{Meas}^+(K)$. For any $k \geq 0$, $\mathcal{M}_k(K)$ denotes the corresponding set of truncated moment sequences: $\mathcal{M}_k(K) := \{(s_0(\mu), \dots, s_k(\mu)) : \mu \in \text{Meas}^+(K)\}$.
- Given an integer $m \geq 1$, a function $F : \mathbb{R}^m \rightarrow \mathbb{R}$ acts on tuples of moment sequences of admissible measures $\mathcal{M}(K_1) \times \dots \times \mathcal{M}(K_m)$ as follows:

$$F[\mathbf{s}(\mu_1), \dots, \mathbf{s}(\mu_m)] := (F(s_k(\mu_1), \dots, s_k(\mu_m)))_{k \geq 0}. \quad (9.1)$$

- Given $h > 0$ and an integer $n \geq 0$, $\Delta_h^n F$ denotes the n th forward difference of the function F with step size h .
- $\mathbf{1}_{m \times n}$ denotes the $m \times n$ matrix with all entries equal to 1.
- $\mathbb{C}^+ := \{z \in \mathbb{C} : \Re z > 0\}$ denotes the right open half-plane.

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